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PRODUCT STRUCTURE OF GRAPH CLASSES WITH STRONGLY SUBLINEAR SEPARATORS

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Abstract. — We investigate the product structure of hereditary graph classes admitting strongly sublinear separators. We characterise such classes as subgraphs of the strong product of a star and a complete graph of strongly sublinear size. In a more precise result, we show that if any hereditary graph class $\mathcal{G}$ admits $O(n^{1-\epsilon})$ separators, then for any fixed $\delta \in (0, \epsilon)$ every n-vertex graph in $\mathcal{G}$ is a subgraph of the strong product of a graph H with bounded tree-depth and a complete graph of size $O(n^{1-\epsilon+\delta})$. This result holds with $\delta = 0$ if we allow H to have tree-depth $O(\log \log n)$. Moreover, using extensions of classical isoperimetric inequalties for grids graphs, we show the dependence on δ in our results and the above $\operatorname{td}(H) \in O(\log \log n)$ bound are both best possible. We prove that n-vertex graphs of bounded treewidth are subgraphs of the product of a graph with tree-depth t and a complete graph of size $O(n^{1/t})$, which is best possible. Finally, we investigate the conjecture that for any hereditary graph class $\mathcal{G}$ that admits $O(n^{1-\epsilon})$ separators, every n-vertex graph in $\mathcal{G}$ is a subgraph of the strong product of a graph H with bounded tree-width and a complete graph of size $O(n^{1-\epsilon})$. We prove this for various classes $\mathcal{G}$ of interest.

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1. Introduction

Graph[(1)] product structure theory describes complicated graphs as subgraphs of strong products[(2)] of simpler building blocks. Examples of graphs classes that can be described this way include planar graphs [38, 79], graphs of bounded Euler genus [30, 38], graphs excluding a fixed minor [18, 38, 58], various non-minor-closed classes [7, 31, 41, 57], and graphs of bounded tree-width [18, 25]. These results have been the key to solving several long-standing open problems about queue layouts [38], nonrepetitive colourings [37], centred colourings [21], adjacency labelling [36, 51], twin-width [16, 59, 61], vertex ranking [17], and box dimension [44]. This paper studies the product structure of graph classes with strongly sublinear separators, which is a more general setting than all of the above classes.

1.1. Background

A *balanced separator* in an n-vertex graph G is a set $S \subseteq V(G)$ such that each component of $G - S$ has at most $\frac{n}{2}$ vertices. The *separation-number* $\operatorname{sep}(G)$ of a graph G is the minimum size of a balanced separator in G.

For a graph parameter f and graph class $\mathcal{G}$, let $f(\mathcal{G})$ be the function $n \mapsto \max\{f(G) : G \in \mathcal{G}, |V(G)| = n\}$. We say $\mathcal{G}$ has *strongly sublinear* f if $f(\mathcal{G}) \in O(n^{1-\epsilon})$ for some fixed $\epsilon > 0$ (where n is always the number of vertices).

Many classes of graphs have strongly sublinear separation-number. For example, Lipton and Tarjan [66] proved that planar graphs have separation-number $O(n^{1/2})$. More generally, Djidjev [33] and Gilbert, Hutchinson and

(1) We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. A *graph class* is a collection of graphs closed under isomorphism. A graph class is *hereditary* if it is closed under taking induced subgraphs. A graph class is *monotone* if it is closed under taking subgraphs. A graph H is *contained* in a graph G if H is isomorphic to a subgraph of G. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A graph G is *H-minor-free* if H is not a minor of G. A graph class $\mathcal{G}$ is *minor-closed* if every minor of each graph in $\mathcal{G}$ is also in $\mathcal{G}$. A *K_h-model* in a graph G consists of pairwise-disjoint vertex-sets $(U_1, \dots, U_h)$ such that, $G[U_i]$ is connected for each i and $G[U_i \cup U_j]$ is connected for all i, j. Clearly K_h is a minor of a graph G if and only if G contains a K_h-model. See [23] for graph-theoretic definitions not given here. The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that G embeds in a surface of Euler genus g; see [70] for more about graph embeddings in surfaces.

(2) The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

Tarjan [54] independently proved that graphs embeddable on any fixed surface have separation-number $O(n^{1/2})$. More generally still, Alon, Seymour and Thomas [2] proved that any proper minor-closed class has separation-number $O(n^{1/2})$. Many non-minor-closed classes also have strongly sublinear separation-number. For example, Grigoriev and Bodlaender [55] proved that graphs that have a drawing in the plane with a bounded number of crossings per edge have separation-number $O(n^{1/2})$. And Miller, Teng, Thurston and Vavasis [69] proved that touching graphs of d-dimensional spheres have separation-number $O(n^{1-1/d})$ (amongst more general results).

The following characterisation of graph classes with strongly sublinear separation-number in terms of tree-width[(3)], path-width and tree-depth[(4)] is folklore.

THEOREM 1.1. — *For fixed $\epsilon \in (0,1)$, the following are equivalent for any hereditary graph class $\mathcal{G}$:*

(a) *$\mathcal{G}$ has separation-number* $\operatorname{sep}(\mathcal{G}) \in O(n^{1-\epsilon})$,
(b) *$\mathcal{G}$ has tree-width* $\operatorname{tw}(\mathcal{G}) \in O(n^{1-\epsilon})$,
(c) *$\mathcal{G}$ has path-width* $\operatorname{pw}(\mathcal{G}) \in O(n^{1-\epsilon})$,
(d) *$\mathcal{G}$ has tree-depth* $\operatorname{td}(\mathcal{G}) \in O(n^{1-\epsilon})$.

Proof. — It follows from the definitions that for every graph G,

$$\operatorname{tw}(G) \leqslant \operatorname{pw}(G) \leqslant \operatorname{td}(G) - 1.$$

Thus (d) ⇒ (c) ⇒ (b). Robertson and Seymour [74, (2.6)] showed that for every graph G,

$$\operatorname{sep}(G) \leqslant \operatorname{tw}(G) + 1. \tag{1.1}$$

Thus (b) ⇒ (a). It is folklore that (a) ⇒ (d); see [3, 10] for proofs that (a) ⇒ (c) which are easily adapted to show that (a) ⇒ (d). (Note that Dvořák and

(3) For a tree T, a *T-decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a (connected) subtree of T. Each set W_x in $\mathcal{W}$ is called a *bag*. The *width* of $\mathcal{W}$ is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *tree-width* $\operatorname{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Tree-width is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has tree-width 1 if and only if it is a tree. A *path-decomposition* is a P-decomposition for any path P. The *path-width* $\operatorname{pw}(G)$ of a graph G is the minimum width of a path-decomposition of G.

(4) A forest is *rooted* if each component has a nominated *root* vertex (which defines the ancestor relation). The *vertex-height* of a rooted forest is the maximum number of vertices in a root–leaf path. The *closure* of a rooted forest F is the graph G with $V(G) := V(F)$ with $vw \in E(G)$ if and only if v is an ancestor of w or vice versa. The *tree-depth* $\operatorname{td}(G)$ of a graph G is the minimum vertex-height of a rooted forest T such that G is a subgraph of the closure of T.

Norin [47] proved a stronger relationship between balanced separators and tree-width that holds without the "strongly sublinear" assumption.) □

We only consider products of the form $H \boxtimes K_m$, which is the "complete blow-up" of the graph H, obtained by replacing each vertex of H by a copy of the complete graph K_m and each edge of H by a copy of the complete bipartite graph $K_{m,m}$. Such products can be characterised via the following definition. For graphs H and G, an *H-partition* of G is a partition $\mathcal{P} = (V_x : x \in V(H))$ of $V(G)$ indexed by the vertices of H such that for each edge $vw \in E(G)$, if $v \in V_x$ and $w \in V_y$ then $xy \in E(H)$ or $x = y$. The *width* of such an H-partition is $\max\{|V_x| : x \in V(H)\}$. The following observation follows immediately from the definitions.

Observation 1.2. — *For any graph H and $m \in \mathbb{N}$, a graph G is contained in $H \boxtimes K_m$ if and only if G has an H-partition of width m.*

If $\mathcal{P}$ is an H-partition of a graph G, where H is a tree or a star, then $\mathcal{P}$ is respectively called a *tree-* or a *star-partition*.[5] The *tree-partition-width* $\text{tpw}(G)$ is the minimum width of a tree-partition of G, and the *star-partition-width* $\text{spw}(G)$ is the minimum width of a star-partition of G. By Observation 1.2, $\text{tpw}(G) \leqslant k$ if and only if G is contained in $T \boxtimes K_k$ for some tree T, and $\text{spw}(G) \leqslant k$ if and only if G is contained in $S \boxtimes K_k$ for some star S.

1.2. Bounded Width Partitions

Our first result (proved in Section 2) characterises graph classes with strongly sublinear separation-number in terms of star- and tree-partition-width.

Theorem 1.3. — *The following are equivalent for any hereditary graph class $\mathcal{G}$:*

(i) *$\mathcal{G}$ has strongly sublinear separation-number,*
(ii) *$\mathcal{G}$ has strongly sublinear tree-partition-width,*

[5] Tree-partitions were independently introduced by Seese [76] and Halin [56], and have since been widely investigated [11, 12, 13, 18, 25, 26, 32, 48, 80, 81]. Tree-partition-width has also been called *strong tree-width* [12, 76]. Applications of tree-partitions include graph drawing [19, 22, 39, 42, 83], nonrepetitive graph colouring [5], clustered graph colouring [1], monadic second-order logic [62], size Ramsey numbers [34], network emulations [8, 9, 14, 52], machine learning theory [84], and the edge-Erdős-Pósa property [20, 53, 72]. Planar-partitions and other more general structures have also been studied [24, 27, 28, 73, 83].

(iii) *$\mathcal{G}$ has strongly sublinear star-partition-width.*

We emphasise that ϵ is fixed in Theorem 1.1, but not in Theorem 1.3. Indeed, there can be a significant difference between the exponents in the bounds on the separator-number and the tree- or star-partition-width in Theorem 1.3. Call this the "*exponent-gap*". For example, while planar graphs have separation-number $\Theta(n^{1/2})$, in Section 2 we show that $\Theta(n^{2/3})$ is a tight bound on the star-partition-width and the tree-partition-width of planar graphs. In this case, the exponent-gap is $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

This motivates the question: can the exponent-gap be reduced by considering products $H \boxtimes K_m$ for graphs H that are more complicated than stars or trees, but still with bounded tree-width or bounded tree-depth? In particular, can the exponent-gap be 0?

Illingworth, Scott and Wood [58] achieved an exponent-gap of 0 for minor-closed graph classes where H has bounded tree-width:

Theorem 1.4 ([58]).

(a) *Every n-vertex planar graph is contained in $H \boxtimes K_m$, for some graph H with $\mathrm{tw}(H) \leqslant 3$, where $m \leqslant \sqrt{8n}$.*
(b) *Every n-vertex graph of Euler genus g is contained in $H \boxtimes K_m$, for some graph H with $\mathrm{tw}(H) \leqslant 3$, where $m \leqslant 4\sqrt{(g+1)n}$.*
(c) *Every n-vertex $K_{s,t}$-minor-free graph is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{tw}(H) \leqslant s$, where $m \leqslant \sqrt{(s-1)(t-1)n}$.*
(d) *Every n-vertex K_t-minor-free graph is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{tw}(H) \leqslant t-2$, where $m \leqslant 2\sqrt{(t-3)n}$.*

In all these results the dependence on n is best possible because $\mathrm{tw}(H \boxtimes K_m) \leqslant (\mathrm{tw}(H)+1)m - 1$ and the $n^{1/2} \times n^{1/2}$ grid has tree-width $n^{1/2}$.

Can similar results be obtained for an arbitrary graph class with strongly sublinear separation-number? The short answer is "almost", as expressed in the next theorem which shows that the exponent-gap can be made arbitrarily small.

Theorem 1.5. — *For $\delta, \epsilon \in \mathbb{R}$ with $0 < \delta < \epsilon < 1$ there exists $t \in \mathbb{N}$ with the following property. Let $\mathcal{G}$ be any hereditary graph class with $\mathrm{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$ for some constant $c > 0$. Then every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{td}(H) \leqslant t$, where $m \leqslant \frac{c\,2^{\epsilon}}{2^{\epsilon}-1}\, n^{1-\epsilon+\delta}$.*

In Theorem 1.5 the graph H has bounded tree-depth. This setting generalises bounded star-partition-width since a graph has tree-depth 2 if and only if it is a star (plus isolated vertices).

Theorem 1.5 is proved in Section 3, where we also prove a result (Theorem 3.2) in which the exponent-gap tends to 0 as $n \to \infty$, at the expense of allowing $\mathrm{td}(H)$ to increase (arbitrarily slowly) with n. Another result (Theorem 3.3) has exponent gap 0 and $\mathrm{td}(H) \in O(\log\log n)$.

In Section 4 we prove various lower bounds that show that many of the results in Section 3 are best possible for multi-dimensional grid graphs. In particular, we show that the dependence on δ in Theorem 1.5 is best possible (Theorem 4.3), and that the above $\mathrm{td}(H) \in O(\log\log n)$ bound is best possible (Theorem 4.4).

It is open whether exponent gap 0 and $\mathrm{tw}(H) \in O(1)$ can be achieved simultaneously.

Open Problem 1.6. — *For any hereditary graph class $\mathcal{G}$ with separation-number $\mathrm{sep}(\mathcal{G}) \in O(n^{1-\epsilon})$, does there exist a constant $c = c(\mathcal{G})$ such that every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$, where $\mathrm{tw}(H) \leqslant c$ and $m \in O(n^{1-\epsilon})$?*

Theorem 1.4 solves Open Problem 1.6 for minor-closed classes with $\epsilon = \frac{1}{2}$. It may even be true that c in Open Problem 1.6 is a function of ϵ only, which was recently proved for minor-closed classes by Distel, Dujmović, Eppstein, Hickingbotham, Joret, Micek, Morin, Seweryn and Wood [29], who also gave improved tree-width bounds for $K_{3,t}$-minor-free graphs, which includes planar and graphs with bounded Euler genus.

Theorem 1.7 ([29]).

(a) *Every n-vertex planar graph is contained in $H \boxtimes K_m$, where $\mathrm{tw}(H) \leqslant 2$ and $m \in O(n^{1/2})$.*
(b) *Every n-vertex graph of Euler genus g is contained in $H \boxtimes K_m$, where $\mathrm{tw}(H) \leqslant 2$ and $m \in O(gn^{1/2})$.*
(c) *Every n-vertex $K_{3,t}$-minor-free graph is contained in $H \boxtimes K_m$, where $\mathrm{tw}(H) \leqslant 2$ and $m \in O(tn^{1/2})$.*
(d) *Every n-vertex K_t-minor-free graph is contained in $H \boxtimes K_m$, where $\mathrm{tw}(H) \leqslant 4$ and $m \in O_t(n^{1/2})$.*

Open Problem 1.6 has an affirmative answer with $c = 1$ if $\mathcal{G}$ has bounded degree, since every graph G with treewidth less than k and maximum degree Δ is contained in $T \boxtimes K_{18k\Delta}$ for some tree T (with maximum degree at most 6Δ) [32].

Section 6 presents several natural graph classes where there is an affirmative answer to Open Problem 1.6, and several natural graph classes where Open Problem 1.6 is unsolved.

1.3. Bounded Treewidth Graphs

Our final contribution concerns the product structure of graphs of bounded treewidth. Given that H has bounded tree-depth in Theorem 1.5, it is natural to consider the following question: Given $k, n, t \in \mathbb{N}$ what is the minimum value of $m = m(k, n, t)$ such that every n-vertex graph with treewidth k is a subgraph of $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant t$. We prove that $m(k, n, t) \in \Theta(n^{1/t})$ for fixed k and t. The following theorem provides the upper bound.

THEOREM 1.8. — *For all $t \in \mathbb{N}$ and $k, n \in \mathbb{N}$, every n-vertex graph G with $\operatorname{tw}(G) \leqslant k$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant t$, where $m \leqslant (k + 1)^{1-1/t} n^{1/t}$.*

The proof of Theorem 1.8 is based on a separator lemma for graphs of given treewidth that generalises (1.1) and is of independent interest (Theorem 5.3). The lower bound, $m(k, n, t) \in \Omega(n^{1/(t+1)})$, follows by considering the case when G is a path. Both these proofs are presented in Section 5.

Loosely speaking, Theorem 1.8 gives good bounds for graphs of bounded treewidth (or very small treewidth as a function of n), whereas Theorem 1.5 (and the other results in Section 3) give better bounds in the strongly sublinear treewidth setting.

2. Star Partitions

Recall Theorem 1.3, which shows that graph classes with strongly sublinear separation-number can be characterised via tree-partitions and star-partitions.

THEOREM 1.3. — *The following are equivalent for any hereditary graph class $\mathcal{G}$:*

(i) *$\mathcal{G}$ has strongly sublinear separation-number,*
(ii) *$\mathcal{G}$ has strongly sublinear tree-partition-width,*
(iii) *$\mathcal{G}$ has strongly sublinear star-partition-width.*

We now prove Theorem 1.3. Let (iv) be the statement that $\mathcal{G}$ has strongly sublinear tree-width. It follows from the definitions that $\operatorname{tpw}(G) \leqslant \operatorname{spw}(G)$ for every graph G; thus (iii) $\Rightarrow$ (ii). Seese [76] observed that $\operatorname{tw}(G) \leqslant 2 \operatorname{tpw}(G) - 1$; thus (ii) $\Rightarrow$ (iv). By Theorem 1.1, (iv) $\Rightarrow$ (i).

It remains to prove that (i) $\Rightarrow$ (iii). Note that any graph G has $\mathrm{spw}(G) \leqslant k$ if and only if G has a set S of at most k vertices such that each component of $G - S$ has at most k vertices. We thus use the following foklore result[(6)].

Lemma 2.1 ([49]). — *Let $\mathcal{G}$ be any hereditary graph class with $\mathrm{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$, for some $c > 0$ and $\epsilon \in (0,1)$. Then for any $\alpha \in (0,1)$ and any n-vertex graph $G \in \mathcal{G}$ there exists $S \subseteq V(G)$ of size at most $\frac{c2^\epsilon}{2^\epsilon - 1} n^{1-\alpha\epsilon}$ such that each component of $G - S$ has at most n^α vertices.*

The next result follows from Lemma 2.1 with $\alpha = \frac{1}{1+\epsilon}$ (since $1 - \alpha\epsilon = 1 - \frac{\epsilon}{1+\epsilon} = \frac{1}{1+\epsilon}$).

Corollary 2.2. — *Let $\mathcal{G}$ be any hereditary graph class with $\mathrm{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$, for some $c > 0$ and $\epsilon \in (0,1)$. Then for every n-vertex graph $G \in \mathcal{G}$ there exists $S \subseteq V(G)$, such that S and each component of $G - S$ has at most $\max\{\frac{c2^\epsilon}{2^\epsilon-1}, 1\} n^{1/(1+\epsilon)}$ vertices, implying*

$$\mathrm{spw}(G) \leqslant \max\{\tfrac{c\,2^\epsilon}{2^\epsilon-1}, 1\}\, n^{1/(1+\epsilon)}.$$

Corollary 2.2 shows that (i) $\Rightarrow$ (iii) in Theorem 1.3, which completes the proof of Theorem 1.3.

As a concrete example, if $\mathcal{G}$ is any hereditary graph class with $\mathrm{sep}(\mathcal{G}) \in O(n^{1/2})$, then $\mathrm{spw}(\mathcal{G}) \in O(n^{2/3})$. For example, for every n-vertex planar graph G,

$$\mathrm{tpw}(G) \leqslant \mathrm{spw}(G) \in O(n^{2/3}).$$

We now show this bound is tight. Consider a graph G with $\mathrm{tpw}(G) \leqslant k$. A proper 2-colouring of the underlying tree determines an improper 2-colouring of G such that each monochromatic component has at most k vertices. Linial, Matoušek, Sheffet and Tardos [65] described an infinite class $\mathcal{G}$ of planar graphs, such that every 2-colouring of any n-vertex graph in $\mathcal{G}$ has a monochromatic component of order $\Omega(n^{2/3})$. So the $O(n^{2/3})$ upper bound on the tree-partition-width of planar graphs is tight.

To conclude this section, we show that the bound on the star-partition-width in Corollary 2.2 is tight for grid graphs. For any integer $d \geqslant 2$, let G^d_n be the d-dimensional $n^{1/d} \times \cdots \times n^{1/d}$ grid graph (where $n^{1/d} \in \mathbb{N}$). Our starting point is the following isoperimetric inequality by Bollobás

(6) Lemma 2.1 is proved by the following argument: initialise $S \coloneqq \varnothing$, while there is a component X of $G - S$ with more than n^α vertices, add a balanced separator in X to S, and repeat until every component of $G - S$ has at most n^α vertices. The idea is present in the work of Lipton and Tarjan [66, 67] for planar graphs, and in the work of Esperet and Raymond [49] and Dvořák and Norin [46] for hereditary graph class with strongly sublinear separators. See [82] for an explicit proof of Lemma 2.1.

and Leader [15, Theorem 3]. For a graph G and $A \subseteq V(G)$, let $\partial(A)$ be the number of edges in G between A and $V(G) \setminus A$.

Lemma 2.3 ([15]). — *For any set A of vertices in G^d_n with $|A| \leqslant \frac{n}{2}$,*

$$\partial(A) \geqslant \min\{|A|^{1-1/r} r\, n^{1/r-1/d} : r \in \{1, \dots, d\}\}.$$

Corollary 2.4. — *For any set A of vertices in G^d_n with $|A| \leqslant \frac{n}{e^d}$,*

$$\partial(A) \geqslant d\,|A|^{1-1/d}.$$

Proof. — By Lemma 2.3 it suffices to show that if $|A| \leqslant \frac{n}{e^d}$ and $r \in \{1, \dots, d\}$, then

$$|A|^{1-1/r} r n^{1/r-1/d} \geqslant d|A|^{1-1/d}. \tag{2.1}$$

Since (2.1) holds with equality when $r = d$, we may assume that $r \neq d$. Let $c := \left(\frac{|A|}{n}\right)^{1/d}$ where $0 \leqslant c \leqslant 1$, and let $x := \frac{r}{d}$ where $0 < x < 1$. Then (2.1) is equivalent to $x^{x/(1-x)} \geqslant c$. The function $x^{x/(1-x)}$ is decreasing when $x \in [0, 1)$, and $\lim_{x \to 1} x^{x/(1-x)} = \frac{1}{e}$. Thus (2.1) holds if $c \leqslant \frac{1}{e}$; that is, if $|A| \leqslant \frac{n}{e^d}$. □

Lemma 2.5. — *If S is any set of at most $\frac{n}{2}$ vertices in G^d_n and $q \leqslant \frac{n}{e^d}$ and each component of $G - S$ has at most q vertices, then $|S| \geqslant \frac{n}{4q^{1/d}}$.*

Proof. — Let $A_1, \dots, A_r$ be the components of $G^d_n - S$. By Corollary 2.4 and since G^d_n has maximum degree $2d$,

$$\begin{aligned} 2d|S| \geqslant \sum_i \partial(A_i) \geqslant \sum_i d|A_i|^{1-1/d} \geqslant \sum_i d|A_i| q^{-1/d} &= dq^{-1/d}(n - |S|) \\ &\geqslant \tfrac{1}{2}(dq^{-1/d} n)\ . \end{aligned}$$

The result follows. □

Now consider the star-partition-width, $\operatorname{spw}(G^d_n)$. If $\operatorname{spw}(G^d_n) = s$ then G^d_n has a set S of at most s vertices such that each component of $G^d_n - S$ has at most s vertices. If $s \leqslant \frac{n}{e^d}$, then $s \geqslant \frac{n}{4s^{1/d}}$ by Lemma 2.5, and thus $s \geqslant (\frac{n}{4})^{d/(d+1)}$. Hence $\operatorname{spw}(G^d_n) \geqslant (\frac{n}{4})^{d/(d+1)}$ when $n \geqslant \frac{e^{d(d+1)}}{4^d}$.

Let $\mathcal{G}^d$ be the class of all subgraphs of d-dimensional grid graphs. Then $\operatorname{sep}(\mathcal{G}^d) \leqslant cn^{1-1/d}$ for some $c = c(d)$ by a result of Miller *et al.* [69]. Thus Corollary 2.2 with $\epsilon = \frac{1}{d}$ proves $\operatorname{spw}(\mathcal{G}^d) \leqslant \max\{\frac{c\,2^{1/d}}{2^{1/d}-1}, 1\}\, n^{d/(d+1)}$, which matches the above lower bound. That is, for fixed d,

$$\operatorname{spw}(\mathcal{G}^d) = \Theta(n^{d/(d+1)}).$$

3. Bounded Tree-depth Partitions

This section shows that n-vertex graphs with strongly sublinear separation-number are contained in $H \boxtimes K_m$ where $\operatorname{td}(H)$ is bounded or is at most a slowly growing function of n, and m is strongly sublinear with respect to n. All the results follow from the next lemma.

Lemma 3.1. — *Let $\mathcal{G}$ be any hereditary graph class with separation-number $\operatorname{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$, for some $c \geqslant 1$ and $\epsilon \in (0,1)$. Then for every $t \in \mathbb{N}$, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant t$, where $m \leqslant \frac{c2^\epsilon}{2^\epsilon-1} n^{(1-\epsilon)/(1-\epsilon^t)}$.*

Proof. — We proceed by induction on t. If $t = 1$ then the claim is trivial—just take $H = K_1$. Now assume that $t \geqslant 2$ and the result holds for $t-1$. Let $\alpha := (1-\epsilon^{t-1})/(1-\epsilon^t) \in (0,1)$ and $\gamma := \frac{c2^\epsilon}{2^\epsilon-1}$. Note that

$$1 - \alpha\epsilon = 1 - \frac{\epsilon(1-\epsilon^{t-1})}{1-\epsilon^t} = \frac{1-\epsilon^t-\epsilon(1-\epsilon^{t-1})}{1-\epsilon^t} = \frac{1-\epsilon}{1-\epsilon^t}.$$

Let G be any n-vertex graph in $\mathcal{G}$. By Lemma 2.1, there exists $S \subseteq V(G)$ of size at most

$$\gamma n^{1-\alpha\epsilon} = \gamma n^{(1-\epsilon)/(1-\epsilon^t)}$$

such that each component of $G-S$ has at most n^α vertices. Say $G_1,\dots,G_k$ are the components of $G-S$. Let $n_i := |V(G_i)| \leqslant n^\alpha$. By induction, G_i is contained in $H_i \boxtimes K_q$ for some graph H_i with $\operatorname{td}(H_i) \leqslant t-1$, where

$$q \leqslant \gamma n_i^{(1-\epsilon)/(1-\epsilon^{t-1})} \leqslant \gamma n^{\alpha(1-\epsilon)/(1-\epsilon^{t-1})} = \gamma n^{(1-\epsilon)/(1-\epsilon^t)}.$$

So H_i is a subgraph of the closure of a rooted tree T_i of vertex-height $t-1$. Let T be obtained from the disjoint union of $T_1,\dots,T_k$ by adding one root vertex r adjacent to the roots of $T_1,\dots,T_k$. Let H be the closure of T. So $\operatorname{td}(H) \leqslant t$. By construction, G is contained in $H \boxtimes K_m$ where $m \leqslant \max\{|S|, q\} \leqslant \gamma n^{(1-\epsilon)/(1-\epsilon^t)}$, as desired. □

Note that Theorem 4.3 in Section 4 shows that the $(1-\epsilon)/(1-\epsilon^t)$ term in Lemma 3.1 is best possible whenever $\frac{1}{\epsilon}$ is an integer at least 2.

Recall Theorem 1.5 from Section 1.

Theorem 1.5. — *For $\delta, \epsilon \in \mathbb{R}$ with $0 < \delta < \epsilon < 1$ there exists $t \in \mathbb{N}$ with the following property. Let $\mathcal{G}$ be any hereditary graph class with $\operatorname{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$ for some constant $c > 0$. Then every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant t$, where $m \leqslant \frac{c\,2^\epsilon}{2^\epsilon-1}\, n^{1-\epsilon+\delta}$.*

Proof. — Apply Lemma 3.1 with $t := \lceil \log_\epsilon(\frac{\delta}{1-\epsilon+\delta}) \rceil$. Note that $\epsilon^t(1-\epsilon+\delta) \leqslant \delta$, implying $1-\epsilon \leqslant (1-\epsilon)+\delta-\epsilon^t(1-\epsilon+\delta) = (1-\epsilon+\delta)(1-\epsilon^t)$. Thus every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant t$, where $m \leqslant \frac{c2^\epsilon}{2^\epsilon-1} n^{(1-\epsilon)/(1-\epsilon^t)} = \frac{c2^\epsilon}{2^\epsilon-1} n^{1-\epsilon+\delta}$. □

The next two results allow $\operatorname{td}(H)$ to increase slowly with n.

Theorem 3.2. — *Fix $\epsilon \in (0,1)$. Let $t : \mathbb{N} \to \mathbb{R}^+$ be any function. For any hereditary graph class $\mathcal{G}$ with $\operatorname{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$ for some constant $c > 0$, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant \lceil t(n) \rceil$, where $m \leqslant \frac{c\,2^\epsilon}{2^\epsilon-1} n^{1-\epsilon+\delta(n)}$ and $\delta(n) := (1-\epsilon)\big((1-\epsilon^{t(n)})^{-1}-1\big)$.*

Proof. — Apply Lemma 3.1 with $t := \lceil t(n) \rceil$. Note that $\delta(n) \geqslant (1-\epsilon) \times (\frac{1}{1-\epsilon^t}-1) = \frac{1-\epsilon}{1-\epsilon^t} - (1-\epsilon)$, implying $\frac{1-\epsilon}{1-\epsilon^t} \leqslant 1-\epsilon+\delta(n)$. Thus every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant \lceil t(n) \rceil$, where $m \leqslant \frac{c\,2^\epsilon}{2^\epsilon-1} n^{(1-\epsilon)/(1-\epsilon^t)} \leqslant \frac{c\,2^\epsilon}{2^\epsilon-1} n^{1-\epsilon+\delta(n)}$. □

The novelty of Theorem 3.2 is that $t(n)$ can be chosen to be any slow-growing function, but the exponent-gap $\delta(n)$ goes to zero as $n \to \infty$. The next result with exponent-gap 0 follows from Theorem 3.2 by taking a specific function h.

Theorem 3.3. — *Fix $\epsilon \in (0,1)$ and $c > 0$. For any hereditary graph class $\mathcal{G}$ with $\operatorname{sep}(\mathcal{G}) \leqslant cn^{1-\epsilon}$, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with $\operatorname{td}(H) \leqslant \lceil \frac{\log(1+\log n)}{-\log \epsilon} \rceil$, where $m \leqslant \frac{2c}{2^\epsilon-1} n^{1-\epsilon}$.*

Proof. — Let $t(n) := \frac{\log(1+\log n)}{-\log \epsilon} = \log_\epsilon(\frac{1}{1+\log n})$. Thus $\epsilon^{t(n)} = \frac{1}{1+\log n} = 1 - \frac{\log n}{1+\log n}$, implying $(1-\epsilon^{t(n)})^{-1} - 1 = (\frac{1+\log n}{\log n}) - 1 = \frac{1}{\log n}$. Hence $\delta(n)\log n = (1-\epsilon)\big((1-\epsilon^{t(n)})^{-1}-1\big)\log n = 1-\epsilon$ and $n^{\delta(n)} = 2^{1-\epsilon}$. The result follows from Theorem 3.2 since $m \leqslant \frac{c\,2^\epsilon}{2^\epsilon-1} n^{1-\epsilon+\delta(n)} = \frac{c\,2^\epsilon 2^{1-\epsilon}}{2^\epsilon-1} n^{1-\epsilon} = \frac{2c}{2^\epsilon-1} n^{1-\epsilon}$. □

4. Lower Bounds

This section proves lower bounds that show that several results in the previous section are best possible. Recall that G^d_n is the d-dimensional $n^{1/d} \times \cdots \times n^{1/d}$ grid graph.

Lemma 4.1. — *Fix integers $d, t \geqslant 2$. Let $G := G^d_n$, let H be any graph with $\operatorname{td}(H) \leqslant t$, and let $A \subseteq V(G)$ such that $G[A]$ has an H-partition of* *width s where*

$$2^{d^{t-1}} \leqslant s \leqslant \left(\tfrac{n}{36e^d}\right)^{(d-1)d^{t-2}/(d^{t-1}-1)}. \tag{4.1}$$

Then

$$\partial(A) \geqslant \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}|A| - 3ds.$$

Proof. — We proceed by induction on t (with d fixed). First suppose that $t = 2$. Let R be the root part in the H-partition of $G[A]$. Thus $|R| \leqslant s$ and $\partial(A - R) \leqslant \partial(A) + \partial(R) \leqslant \partial(A) + 2ds$ (since G has maximum degree at most $2d$). Let C be the vertex-set of a component of $G[A] - R$. Thus $|C| \leqslant s \leqslant \frac{n}{e^d}$ and $\partial(C) \geqslant d|C|^{1-1/d}$ by Corollary 2.4. Summing over all such C,

$$\begin{aligned}\partial(A - R) = \sum_C \partial(C) \geqslant \sum_C d|C|^{1-1/d} \geqslant \sum_C d|C|s^{-1/d} &= ds^{-1/d}(|A| - |R|)\\ &\geqslant ds^{-1/d}(|A| - s).\end{aligned}$$

Thus

$$\partial(A) \geqslant \partial(A - R) - 2ds \geqslant ds^{-1/d}(|A| - s) - 2ds > \tfrac{d}{6}\, s^{-1/d}|A| - 3ds,$$

as claimed.

Now assume that $t \geqslant 3$ and the result holds for $t-1$. Let $A \subseteq V(G)$ such that $G[A]$ has an H-partition of width s, for some graph H with $\mathrm{td}(H) \leqslant t$. Let R be the root part in the H-partition of $G[A]$. Thus $|R| \leqslant s$ and

$$\partial(A - R) \leqslant \partial(A) + \partial(R) \leqslant \partial(A) + 2ds. \tag{4.2}$$

Let C be the vertex-set of a component of $G[A] - R$. So $G[C]$ has an H'-partition of width s, for some graph H' with $\mathrm{td}(H') \leqslant t - 1$. Say C is *big* if $|C| \geqslant 36\, s^{(d^{t-1}-1)/((d-1)d^{t-2})}$ and *small* otherwise.

First suppose that C is small. Then $|C| \leqslant 36\, s^{(d^{t-1}-1)/((d-1)d^{t-2})}$, which is at most $\frac{n}{e^d}$ by the upper bound on s in Eq. (4.1). Thus Corollary 2.4 is applicable, and

$$\begin{aligned}\partial(C) \geqslant d|C|^{1-1/d} &\geqslant \tfrac{d}{6}\,|C|\left(s^{(d^{t-1}-1)/((d-1)d^{t-2})}\right)^{-1/d}\\ &= \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\,|C|.\end{aligned}$$

Now suppose that C is big. Since Eq. (4.1) holds for t, it also holds for $t - 1$. Thus, by induction,

$$\begin{aligned}\partial(C) \geqslant \tfrac{d}{6}\, s^{-(d^{t-2}-1)/((d-1)d^{t-2})}|C| - 3ds &\geqslant \tfrac{d}{12}\, s^{-(d^{t-2}-1)/((d-1)d^{t-2})}|C|\\ &\geqslant \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\,|C|,\end{aligned}$$

where the final inequality holds since $s \geqslant 2^{d^{t-1}}$. Summing over all such components,

$$\begin{aligned}\partial(A-R) = \sum_C \partial(C) &\geqslant \sum_C \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\, |C| \\ &= \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\, (|A|-|R|) \\ &\geqslant \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\, (|A|-s).\end{aligned}$$

By Eq. (4.2),

$$\begin{aligned}\partial(A) \geqslant \partial(A-R) - 2ds &\geqslant \tfrac{d}{6}\, s^{-(d^{t-1}-1)/((d-1)d^{t-1})}\, (|A|-s) - 2ds \\ &\geqslant \tfrac{d}{6}\, s^{(d^{t-1}-1)/((d-1)d^{t-1})}\, |A| - 3ds,\end{aligned}$$

as desired. □

Lemma 4.2. — *Fix integers $d, t \geqslant 2$ and $s \geqslant 5^{d^t}$. Let $G := G_n^d$ where $n \gg d, t$. Let H be any graph with $\operatorname{td}(H) \leqslant t$, such that G has an H-partition of width s. Then*

$$s \geqslant \left(\tfrac{n}{12}\right)^{(d-1)d^{t-1}/(d^t-1)}.$$

Proof. — If $s \geqslant \left(\frac{n}{36e^d}\right)^{(d-1)d^{t-2}/(d^{t-1}-1)}$ then $s \geqslant \left(\frac{n}{12}\right)^{(d-1)d^{t-1}/(d^t-1)}$ for large enough $n \gg d, t$, and we are done. Now assume that $s \leqslant \left(\frac{n}{36e^d}\right)^{(d-1)d^{t-2}/(d^{t-1}-1)}$, which is required below when applying Lemma 4.1.

Let M be the root part in the H-partition of G. Let $A_1, \dots, A_p$ be the vertex-sets of the components of $G - M$.

First suppose that $t = 2$. Thus $|M|, |A_1|, \dots, |A_p| \leqslant s \leqslant \frac{n}{e^d}$. By Lemma 2.5, $s \geqslant |M| \geqslant \frac{n}{4s^{1/d}}$ and $4s^{(d+1)/d} \geqslant n$ and

$$s \geqslant \left(\tfrac{n}{4}\right)^{d/(d+1)} \geqslant \left(\tfrac{n}{12}\right)^{(d-1)d/(d^2-1)},$$

as desired.

Now assume that $t \geqslant 3$. By assumption, each $G[A_i]$ has an H_i-partition of width at most s, for some graph H_i with $\operatorname{td}(H_i) \leqslant t-1$. By Lemma 4.1,

$$\tfrac{d}{6}\, s^{-(d^{t-2}-1)/((d-1)d^{t-2})}|A_i| - 3ds \leqslant \partial(A_i) \leqslant \partial(M) \leqslant 2d|M| \leqslant 2ds,$$

implying

$$|A_i| \leqslant q := 30s^{1+(d^{t-2}-1)/((d-1)d^{t-2})} = 30s^{(d^{t-1}-1)/((d-1)d^{t-2})}.$$

Since $s \geqslant 5^{d^t}$,

$$\tfrac{30}{12}e^d \leqslant 5^d \leqslant s^{1/d^{t-1}} = s^{(d^t-1)/((d-1)d^{t-1}) - (d^{t-1}-1)/((d-1)d^{t-2})},$$

implying

$$30e^d\, s^{(d^{t-1}-1)/((d-1)d^{t-2})} \leqslant 12\, s^{(d^t-1)/((d-1)d^{t-1})}.$$

We may assume the right-hand-side is less than n, otherwise we are done. Thus

$$q = 30s^{(d^{t-1}-1)/((d-1)d^{t-2})} \leqslant \tfrac{n}{e^d}.$$

Hence Corollary 2.4 is applicable to A_i, and

$$\begin{aligned} 2ds \geqslant \partial(M) = \sum_i \partial(A_i) \geqslant \sum_i d|A_i|^{1-1/d} \geqslant \sum_i d|A_i|q^{-1/d} \\ = dq^{-1/d}(n-|M|) \\ \geqslant dq^{-1/d}(n-s). \end{aligned}$$

Therefore

$$\begin{aligned} n \leqslant 2sq^{1/d} + s &= 2s\big(30s^{(d^{t-1}-1)/((d-1)d^{t-2})}\big)^{1/d} + s \\ &= 2\cdot 30^{1/d}\, s^{(d^t-1)/((d-1)d^{t-1})} + s \\ &< 12s^{(d^t-1)/((d-1)d^{t-1})}. \end{aligned}$$

The result follows. □

We now drop the $s \geqslant 5^{d^t}$ assumption in Lemma 4.2.

Theorem 4.3. — *Fix integers $d, t \geqslant 2$. Let $G := G_n^d$ where $n \gg d, t$. For any graph H with $\operatorname{td}(H) \leqslant t$, if G has an H-partition of width s, then*

$$s \geqslant \left(\tfrac{n}{12}\right)^{(d-1)d^{t-1}/(d^t-1)}.$$

Proof. — By Lemma 4.2 we may assume that $s < 5^{d^t}$. Then G has an H-partition of width 5^{d^t}. By Lemma 4.2,

$$5^{d^t} \geqslant \left(\tfrac{n}{12}\right)^{(d-1)d^{t-1}/(d^t-1)}.$$

Since $(d-1)d^{t-1}/(d^t-1) > 0$, taking $n \gg d, t$ we obtain a contradiction. □

As mentioned earlier, subgraphs of d-dimensional grids are a hereditary class with separation-number $O(n^{1-1/d})$. The exponent of n in the lower bound in Theorem 4.3 matches the corresponding upper bound in Lemma 3.1 with $\epsilon = \frac{1}{d}$ since

$$(1-\epsilon)/(1-\epsilon^t) = (1-\tfrac{1}{d})/(1-\tfrac{1}{d^t}) = \tfrac{d-1}{d}\cdot\tfrac{d^t}{d^t-1} = \tfrac{(d-1)d^{t-1}}{d^t-1}.$$

Thus Lemma 3.1 is best possible whenever $\frac{1}{\epsilon}$ is an integer at least 2.

To conclude this section, we now show that the $O(\log\log n)$ term in Theorem 3.3 is best possible.

Theorem 4.4. — *Fix any integer $d \geqslant 2$. Assume that there is a function t, such that for some $c > 0$, every d-dimensional grid graph G^d_n is contained in $H \boxtimes K_s$, for some graph H with $\operatorname{td}(H) \leqslant t(n)$, and where $s \leqslant cn^{1-1/d}$. Then $t(n) \in \Omega(\log\log n)$.*

Proof. — Let $G := G^d_n$ and $t := t(n)$. By Observation 1.2, G has an H-partition of width $s = \lfloor cn^{1-1/d} \rfloor$. If $s \leqslant 5^{d^t}$ then $cn^{1-1/d} \leqslant 5^{d^t} + 1$ and $t(n) \in \Omega(\log\log n)$, as desired. Otherwise, $s \geqslant 5^{d^t}$, and by Lemma 4.2,

$$cn^{1-1/d} \geqslant s \geqslant \left(\tfrac{n}{12}\right)^{(d-1)d^{t-1}/(d^t-1)}.$$

Thus

$$12c \geqslant c\,12^{(d-1)d^{t-1}/(d^t-1)} \geqslant n^{(d-1)d^{t-1}/(d^t-1)-1+1/d} = n^{(d-1)/(d(d^t-1))}.$$

Thus $(12c)^{d(d^t-1)/(d-1)} \geqslant n$, and $t(n) \in \Omega(\log\log n)$, as desired. ☐

Note that Theorem 4.4 implies that the strengthening of Open Problem 1.6 with tree-width replaced by tree-depth is false.

5. Bounded Tree-width Graphs

This section considers H-partitions of graphs with bounded tree-width, where H has bounded tree-depth. We start with star-partitions of trees.

Lemma 5.1. — *For any $p, q, n \in \mathbb{N}$ with $n \leqslant pq + p + q$, any n-vertex tree T has a set S of at most p vertices such that each component of $T - S$ has at most q vertices.*

Proof. — We proceed by induction on p. The base case with $p = 1$ is folklore. We include the proof for completeness. Orient each edge vw of G from v to w if the component of $T - vw$ containing v has at most $\lfloor \frac{n}{2} \rfloor$ vertices. Each edge is oriented by this rule. Since T is acyclic, there is a vertex v in T with outdegree 0. So each component of $T - v$ has at most $\lfloor \frac{n}{2} \rfloor \leqslant q$ vertices, and the result holds with $S = \{v\}$.

Now assume $p \geqslant 2$. Root T at an arbitrary vertex r. For each vertex v, let T_v be the subtree of T rooted at v. For each leaf vertex v, let $f(v) = 0$. For each non-leaf vertex v, let $f(v) := \max_w |V(T_w)|$ where the maximum is taken over all children w of v. If $f(r) \leqslant q$ then $S = \{r\}$ satisfies the claim. Now assume that $f(r) \geqslant q+1$. Let v be a vertex of T at maximum distance from r such that $f(v) \geqslant q + 1$. This is well-defined since $f(r) \geqslant q + 1$. By definition, $|V(T_w)| \geqslant q + 1$ for some child w of v, but $f(w) \leqslant q$. So every subtree rooted at a child of w has at most q vertices. Let T' be the subtree of T obtained by deleting the subtree rooted at w. Thus $|V(T')| = n - |V(T_w)| \leqslant n-(q+1) \leqslant pq+p+q-(q+1) = (p-1)q+(p-1)+q$. By induction, T' has a set S' of at most $p-1$ vertices such that each component of $T'-S'$ has at most q vertices. Let $S := S' \cup \{w\}$. By construction, $|S| \leqslant p$ and each component of $T-S$ has at most q vertices. □

Lemma 5.1 also follows from a Helly-type property stated in [75, (8.6)][(7)]. We now show the bound in Lemma 5.1 is best possible for the n-vertex path T. Say T has a set S of p vertices such that each component of $T-S$ has at most q vertices. Since T is a path, $T-S$ has at most $p+1$ components, each with at most q vertices. Thus $n \leqslant (p+1)q+p$.

To generalise Lemma 5.1 for graphs of given tree-width, we need the following normalisation lemma[(8)].

Lemma 5.2. — *Every graph with tree-width k has a tree-decomposition $(B_x : x \in V(T))$ such that:*

(a) $|B_x| = k+1$ *for each* $x \in V(T)$,

(b) *for each edge* $xy \in E(T)$, *we have* $|B_x \setminus B_y| = 1$ *and* $|B_y \setminus B_x| = 1$,

(c) $|V(T)| = |V(G)| - k$, *and*

(d) *for any non-empty set* $S \subseteq V(T)$, *for each component* T' *of* $T-S$,

$$\left| \left(\bigcup_{x \in V(T')} B_x \right) \setminus \left(\bigcup_{x \in S} B_x \right) \right| \leqslant |V(T')|.$$

Proof. — Since G has tree-width k, G has a tree-decomposition $(B_x : x \in V(T))$ such that $|B_x| \leqslant k+1$ for each $x \in V(T)$.

If $|B_x| > |B_y|$ for some edge xy of T, then add one vertex from $B_x \setminus B_y$ to B_y, and repeat this step until $|B_x| = |B_y|$ for each edge $xy \in E(T)$. Now (a) is satisfied (since G has tree-width k).

If $B_x = B_y$ for some edge $xy \in E(T)$, then contract xy into a new vertex z with $B_z := B_x$. This operation maintains that $|B_x| = k+1$ for each $x \in V(T)$. Repeat this operation until $B_x \neq B_y$ for each edge $xy \in E(T)$.

Say $|B_x \setminus B_y| \geqslant 2$ for some edge $xy \in E(T)$. Thus $|B_y \setminus B_x| \geqslant 2$. Let $v \in B_x \setminus B_y$ and $w \in B_y \setminus B_x$. Delete the edge xy from T, and introduce a

(7) Statement (8.6) in [75] says that for any family $\mathcal{F}$ of subtees of a tree T, for any integer $k \geqslant 0$, either (i) there are k vertex-disjoint element of $\mathcal{F}$, or (ii) there is a subset $X \subseteq V(T)$ with $|X| < k$ that intersects every element of $\mathcal{F}$. See [58, Lemma 9] for a proof of this statement. Under the assumptions of Lemma 5.1, apply this result where $\mathcal{F}$ is the family of subtrees of T with at least $q+1$ vertices, and $k := p+1$. If (i) holds, then T has $p+1$ disjoint subtrees, each with $q+1$ vertices, implying $|V(T)| \geqslant (p+1)(q+1) = pq+p+q+1$, which is a contradiction. Outcome (ii) gives the desired set, since $T-X$ contains no subtree of T on at least $q+1$ vertices.

(8) Conditions (a), (b) and (c) in Lemma 5.2 are well-known. Condition (d) may be new.

new vertex z to T only adjacent to x and y, where $B_z := (B_x \setminus \{v\}) \cup \{w\}$. This operation maintains property (a) and still $B_x \neq B_y$ for each edge $xy \in E(T)$. Repeat this operation until (b) is satisfied.

We now prove (c). Let r be any vertex of T. Define a function $f : V(T-r) \to V(G) \setminus B_r$, where for each $y \in V(T-r)$, if p is the neighbour of y in T closest to r, then $f(y)$ is the vertex in $B_y \setminus B_p$. By (b), f is a bijection. Thus $|V(G) \setminus B_r| = |V(T-r)|$ and $|V(G)| = |B_r| + |V(G) \setminus B_r| = (k+1) + (|V(T)| - 1) = k + |V(T)|$. This proves (c).

We now prove (d). Let S be a non-empty subset of $V(T)$. Let T' be a component of $T - S$. Let $V' := (\bigcup_{y \in V(T')} B_y) \setminus (\bigcup_{x \in S} B_x)$. There is a vertex $x \in S$ adjacent to some vertex in T'. For each $y \in V(T')$, if p is the neighbour of y in T closest to x, then let v_y be the vertex in $B_y \setminus B_p$. Then $v_y \neq v_z$ for all distinct $y, z \in V(T')$. Each vertex of V' equals v_y for some $y \in V(T')$. Thus $|V'| \leqslant |V(T')|$. This proves (d). □

Theorem 5.3. — *Let $p, q, k, n \in \mathbb{N}$ with $n \leqslant \lfloor \frac{p}{k+1} \rfloor (q+1) + q + k$ and $p \geqslant k+1$. Then any n-vertex graph G with $\operatorname{tw}(G) \leqslant k$ has a set S of p vertices such that each component of $G - S$ has at most q vertices.*

Proof. — Let $(B_x : x \in V(T))$ be a tree-decomposition of G satisfying Lemma 5.2. Let $p' := \lfloor \frac{p}{k+1} \rfloor$. Thus $p' \geqslant 1$ and $p'(q+1)+q \geqslant n-k = |V(T)|$ by Lemma 5.2(c). By Lemma 5.1, T has a set S_0 of p' vertices such that each component of $T - S_0$ has at most q vertices. Let $S := \bigcup_{x \in S_0} B_x$. So $|S| \leqslant (k+1)|S_0| \leqslant (k+1)p' \leqslant p$. For each component G' of $G - S$ there is a subtree T' of $T - S_0$ such that $V(G') \subseteq (\bigcup_{x \in V(T')} B_x) \setminus S$, implying $|V(G')| \leqslant |V(T')| \leqslant q$ by Lemma 5.2(d). □

Theorem 5.3 with $p = k+1$ and $q = \lceil \frac{n-k-1}{2} \rceil$ says that every graph G with $\operatorname{tw}(G) \leqslant k$ has a set S of $k+1$ vertices such that each component of $G - S$ has at most $q \leqslant \frac{n-k}{2}$ vertices, which implies Eq. (1.1) by Robertson and Seymour [74, (2.6)]. Thus Theorem 5.3 generalises the result of Robertson and Seymour [74] and is of independent interest. Note that Thomassen [78, Proposition 2.1] proved a similar, but less precise, statement to Theorem 5.3.

We now show that Theorem 5.3 is roughly best possible for the k-th power of the path, P_n^k. This graph has vertex-set $\{v_1, \dots, v_n\}$ where $v_i v_j$ is an edge if and only if $|i-j| \in \{1, \dots, k\}$. Note that $\operatorname{tw}(P_n^k) = \operatorname{pw}(P_n^k) = k$. Consider the graph P_n^k for any integer $n > \frac{pq}{k} + p + q$. Let S be any set of p vertices in P_n^k, and define a *block* to be a maximal set of vertices in S that are consecutive with respect to P_n. Let b be the number of blocks of size at least k. So $p \geqslant bk$ and the number of components of $P_n^k - S$ is at most $b+1$ (since if B is a block of size at most $k-1$, then the vertices immediately before and after B are adjacent in P_n^k). Since $n > \frac{pq}{k} + p + q \geqslant (b+1)q + p$, it follows that $P_n^k - S$ has a component with more than q vertices. Hence the bound in Theorem 5.3 cannot be improved to $n \leqslant \frac{pq}{k} + p + q + 1$.

Theorem 5.3 with $p = q = \lceil \sqrt{(\operatorname{tw}(G)+1)n} \rceil$ implies that for every n-vertex graph G,

$$\operatorname{tpw}(G) \leqslant \operatorname{spw}(G) \leqslant \left\lceil \sqrt{(\operatorname{tw}(G)+1)n} \right\rceil.$$

The second inequality here is essentially best possible since the argument above shows that $\operatorname{spw}(P_n^k) = (1+o(1))\sqrt{kn}$. We now show that the same upper bound on $\operatorname{tpw}(G)$ is also essentially best possible. Let $k \geqslant 2$ and let G be the graph obtained from P_n^{k-1} by adding one dominant vertex. So $\operatorname{tw}(G) = k$. In any tree-partition of G, since G has a dominant vertex, the tree is a star. Since P_n^{k-1} is a subgraph of G, we have $\operatorname{tpw}(G) \geqslant (1-o(1))\sqrt{(k-1)n}$.

We now set out to generalise Theorem 5.3 for H-partitions, where H has bounded tree-depth. We need the following analogue of Lemma 5.1 where q is allowed to be real.

Lemma 5.4. — *For any $p, n \in \mathbb{N}$ and $q \in \mathbb{R}^+$ with $n \leqslant q(p+1)$, any n-vertex tree T has a set S of at most p vertices such that each component of $T - S$ has at most q vertices.*

Proof. — We proceed by induction on p. The base case with $p = 1$ is identical to the $p = 1$ case of Lemma 5.1. Now assume that $p \geqslant 2$. Root T at an arbitrary vertex r. For each vertex v, let T_v be the subtree of T rooted at v. For each leaf vertex v, let $f(v) = 0$. For each non-leaf vertex v, let $f(v) := \max_w |V(T_w)|$ where the maximum is taken over all children w of v. If $f(r) \leqslant q$ then $S = \{r\}$ satisfies the claim. Now assume that $f(r) > q$. Let v be a vertex of T at maximum distance from r such that $f(v) > q$. This is well-defined since $f(r) > q$. By definition, $|V(T_w)| > q$ for some child w of v, but $f(w) \leqslant q$. So every subtree rooted at a child of w has at most q vertices. Let T' be the subtree of T obtained by deleting the subtree rooted at w. Thus

$$|V(T')| = n - |V(T_w)| < n - q \leqslant q(p+1) - q = qp.$$

By induction, T' has a set S' of at most $p-1$ vertices such that each component of $T' - S'$ has at most q vertices. Let $S := S' \cup \{w\}$. By construction, $|S| \leqslant p$ and each component of $T - S$ has at most q vertices. ☐

We have the following analogue of Theorem 5.3 where p and q are both allowed to be real.

LEMMA 5.5. — *Let $k, n \in \mathbb{N}$ and $p, q \in \mathbb{R}_{>0}$ with $n(k+1) \leqslant pq$ and $p \geqslant k+1$. Then any n-vertex graph G with $\mathrm{tw}(G) \leqslant k$ has a set S of at most p vertices such that each component of $G - S$ has at most q vertices.*

Proof. — Let $(B_x : x \in V(T))$ be a tree-decomposition of G satisfying Lemma 5.2. Let $p' := \lfloor \frac{p}{k+1} \rfloor$. Thus $p' \in \mathbb{N}$ and $(p'+1)q \geqslant \frac{p}{k+1} q \geqslant n$. By Lemma 5.4, T has a set S_0 of at most p' vertices such that each component of $T - S_0$ has at most q vertices. Let $S := \bigcup_{x \in S_0} B_x$. So $|S| \leqslant (k+1)|S_0| \leqslant (k+1)p' \leqslant p$. For each component G' of $G - S$ there is a subtree T' of $T - S_0$ such that $V(G') \subseteq (\bigcup_{x \in V(T')} B_x) \setminus S$, implying $|V(G')| \leqslant |V(T')| \leqslant q$ by Lemma 5.2(d). □

We now reach the main result of this section (where the case $t = 2$ in Theorem 1.8 is roughly equivalent to Theorem 5.3).

THEOREM 1.8. — *For all $t \in \mathbb{N}$ and $k, n \in \mathbb{N}$, every n-vertex graph G with $\mathrm{tw}(G) \leqslant k$ is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{td}(H) \leqslant t$, where $m \leqslant (k+1)^{1-1/t} n^{1/t}$.*

Proof. — We proceed by induction on t. With $t = 1$, the claim holds with $H = K_1$ and $m = n$. Now assume $t \geqslant 2$ and the claim holds for $t-1$. Without loss of generality, we can assume that $\mathrm{tw}(G) = k$, and thus $n \geqslant k+1$. Let $p := (k+1)^{1-1/t} n^{1/t} \geqslant k+1$ and $q := (k+1)^{1/t} n^{1-1/t}$. So $n(k+1) = pq$. By Lemma 5.5, there is a set S of at most p vertices such that each component of $G - S$ has at most q vertices. Let $G_1, \ldots, G_c$ be the components of $G - S$. Each G_i has treewidth at most k. By induction, G_i is contained in $H_i \boxtimes K_{m'}$ for some graph H_i with $\mathrm{td}(H_i) \leqslant t-1$ and $m' \leqslant (k+1)^{(t-2)/(t-1)} q^{1/(t-1)} = (k+1)^{1-1/t} n^{1/t}$. So H_i is a subgraph of the closure of a rooted tree T_i of vertex-height at most $t-1$. Let T be the rooted tree obtained from the disjoint union of $T_1, \ldots, T_c$ by adding a root vertex r adjacent to the root of each T_i. So $\mathrm{td}(H) \leqslant t$. Let S be the part associated with r. Hence G is contained in $H \boxtimes K_m$ where $m \leqslant \max\{|S|, m'\} \leqslant (k+1)^{1-1/t} n^{1/t}$. □

We now show that the dependence on n in Theorem 1.8 is best possible. In particular, we show that if the n-vertex path is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{td}(H) \leqslant t$, then $m \geqslant \Omega(n^{1/t})$ for fixed t. We proceed by induction on $t \geqslant 1$ with the following hypothesis: if the n-vertex path P is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{td}(H) \leqslant t$, then $n \leqslant (2m)^t$. If $m = 1$ then this says that $\mathrm{td}(P_n) \geqslant \log n$, which holds [71]. Now assume that $m \geqslant 2$. In the base case, if $t = 1$ then $H = K_1$ and $n \leqslant m$ as desired. Now assume that $t \geqslant 2$, and the n-vertex path P is contained in $H \boxtimes K_m$ for some graph H with $\mathrm{td}(H) \leqslant t$. Let S be the set of at most m vertices of

G associated with the root vertex r of H. Since $P - S$ has at most $|S| + 1$ components, some sub-path P' of $P - S$ has $n' \geqslant \frac{n-|S|}{|S|+1} \geqslant \frac{n-m}{m+1}$ vertices. If $\frac{n-m}{m+1} < \frac{n}{2m}$, then $n < \frac{2m^2}{m-1} < (2m)^2 \leqslant (2m)^t$, as desired. Hence, suppose that $n' \geqslant \frac{n-m}{m+1} \geqslant \frac{n}{2m}$. Since P' is connected, P' is contained in $H' \boxtimes K_m$, where H' is some component of $H - r$, which implies $\operatorname{td}(H') \leqslant t - 1$. By induction $\frac{n}{2m} \leqslant n' \leqslant (2m)^{t-1}$. Hence $n \leqslant (2m)^t$, as desired.

6. Regarding Open Problem 1.6

This section presents several results related to Open Problem 1.6. First we describe two methods, shallow minors and weighted separators, that can be used to show that various graph classes satisfy Open Problem 1.6. We then give examples of graphs that highlight the difficulty of Open Problem 1.6. We conclude the paper by presenting several interesting graph classes for which Open Problem 1.6 is unsolved.

6.1. Using Shallow Minors

For any integer $r \geqslant 0$, a graph H is an *r-shallow minor* of a graph G if H can be obtained from G by contracting pairwise-disjoint subgraphs of G, each with radius at most r, and then taking a subgraph. Let $\nabla_r(G)$ be the maximum average degree of an r-shallow minor of G. Shallow minors are helpful for attacking Open Problem 1.6. Hickingbotham and Wood [57] proved the following, where $\Delta(G)$ is the maximum degree of G.

Lemma 6.1 ([57]). — *For any $r \in \mathbb{N}_0$ and $\ell, t \in \mathbb{N}$, for any graphs H and L where $\operatorname{tw}(H) \leqslant t$ and $\Delta(L^r) \leqslant k$, if a graph G is an r-shallow minor of $H \boxtimes L \boxtimes K_\ell$, then G is contained in $J \boxtimes L^{2r+1} \boxtimes K_{\ell(k+1)}$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{2r+1+t}{t} - 1$.*

Lemma 6.1 with $L = K_1$ and $k = 0$ implies:

Corollary 6.2. — *For any graph H with $\operatorname{tw}(H) \leqslant t$, for any $r \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, if a graph G is an r-shallow minor of $H \boxtimes K_\ell$, then G is contained in $J \boxtimes K_\ell$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{2r+1+t}{t} - 1$.*

Corollary 6.2 essentially says that shallow minors of graphs that satisfy Open Problem 1.6 also satisfy Open Problem 1.6. We now give two examples of this approach.

A graph G is *(g, k)-planar* if there is a drawing of G in a surface of Euler genus g with at most k crossings on each edge (assuming no three edges cross at a single point).

PROPOSITION 6.3. — *Every n-vertex (g, k)-planar graph G is contained in $L \boxtimes K_\ell$ for some graph L with $\operatorname{tw}(L) \leqslant \binom{k+5}{3} - 1$, where $\ell \leqslant 8\sqrt{(g+1)(1+k^{3/2}d_g)n}$ and $d_g := \max\{3, \frac{1}{4}(5+\sqrt{24g+1})\}$.*

Proof. — Theorem 1.4(b) establishes the $k = 0$ case. Now assume that $k \geqslant 1$. Fix a drawing of G in a surface of Euler genus g with at most k crossings on each edge. Let C be the total number of crossings. Let $m := |E(G)|$. So $C \leqslant \frac{k}{2}m$. Ossona de Mendez, Oum and Wood [68, Lemma 4.5] proved the following generalisation of the Crossing Lemma: if $m > 2d_g n$ then $C \geqslant \frac{m^3}{8(d_g n)^2}$. Assume for the time being that $m > 2d_g n$. Thus $\frac{m^3}{8(d_g n)^2} \leqslant C \leqslant \frac{k}{2}m$, implying $m^2 \leqslant 4k(d_g n)^2$ and $m \leqslant 2\sqrt{k}d_g n$. Thus $m \leqslant \max\{2d_g, 2\sqrt{k}d_g\}n = 2\sqrt{k}d_g n$. Hence $C \leqslant \frac{k}{2}m \leqslant k^{3/2}d_g n$. Hickingbotham and Wood [57] showed that G is a $\lceil \frac{k}{2} \rceil$-shallow minor of $H \boxtimes K_2$, where H is the graph of Euler genus g obtained from G by adding a vertex at each crossing point. Hence $|V(H)| \leqslant n + C < (1 + k^{3/2}d_g)n$. By Theorem 1.4(b), H is contained in $J \boxtimes K_{\ell'}$, for some graph J with $\operatorname{tw}(J) \leqslant 3$, where $\ell' \leqslant 4\sqrt{(g+1)|V(H)|} \leqslant 4\sqrt{(g+1)(1+k^{3/2}d_g)n}$. Hence G is a $\lceil \frac{k}{2} \rceil$-shallow minor of $J \boxtimes K_{2\ell'}$. By Corollary 6.2 with $t = 3$ and $r = \lceil \frac{k}{2} \rceil$, we have that G is contained in $L \boxtimes K_{2\ell'}$ for some graph L with $\operatorname{tw}(L) \leqslant \binom{k+5}{3} - 1$. □

Here is a second example. A graph G is *fan-planar* if there is a drawing of G in the plane such that for each edge $e \in E(G)$ the edges that cross e have a common end-vertex and they cross e from the same side (when directed away from their common end-vertex) [6, 60].

COROLLARY 6.4. — *Every n-vertex fan-planar graph G is contained in $J \boxtimes K_m$ where $\operatorname{tw}(J) \leqslant 19$ and $m < 29\sqrt{n}$.*

Proof. — Kaufmann and Ueckerdt [60] proved that G has less than $5n$ edges. Hickingbotham and Wood [57] showed that G is a 1-shallow minor of $H \boxtimes K_3$ for some planar graph H with $|V(H)| \leqslant |V(G)| + 2|E(G)| < 11n$. By Theorem 1.4(b), H is contained in $J \boxtimes K_{m'}$, for some graph J with $\operatorname{tw}(J) \leqslant 3$, where $m' \leqslant 2\sqrt{2|V(H)|} \leqslant 2\sqrt{22n} < \frac{29}{3}\sqrt{n}$. Thus G is a 1-shallow minor of $J \boxtimes K_{3m'}$. By Corollary 6.2 with $t = 3$, G is contained in $J \boxtimes K_{3m'}$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{6}{3} - 1 = 19$. □

6.2. Using Weighted Separators

The following definition and lemma provides another way to show that various graph classes satisfy Open Problem 1.6. A graph J is *(n, m)-separable* if for every vertex-weighting of J with non-negative real-valued weights and with total weight n, there is a set $S \subseteq V(J)$ with total weight m, such that each component of $J - S$ has at most m vertices. There are numerous results about weighted separators in the literature, but most of these consider the total weight of each component of $J - S$ instead of the weight of S itself. Such considerations are studied in depth by Dvořák [43].

LEMMA 6.5. — *For any graph H and any (n, m)-separable graph J, if G is any n-vertex graph contained in $H \boxtimes J$, then G is contained in $L \boxtimes K_m$ for some graph L with $\operatorname{tw}(L) \leqslant \operatorname{tw}(H) + 1$.*

Proof. — We may assume that G is a subgraph of $H \boxtimes J$. So each vertex of G is of the form (x, y) where $x \in V(H)$ and $y \in V(J)$. Weight each vertex y of J by the number of vertices $x \in V(H)$ such that (x, y) is in G. So the total weight is n. By assumption, there is a set $S \subseteq V(J)$ with total weight m, where each component of $J - S$ has at most m vertices. Let $J_1, \ldots, J_t$ be the components of $J - S$. Let $H_1, \ldots, H_t$ be disjoint copies of H. Let L be obtained from $H_1 \cup \cdots \cup H_t$ by adding one dominant vertex z. Note that $\operatorname{tw}(L) \leqslant \operatorname{tw}(H) + 1$. For each vertex x of H, let x_i be the copy of x in H_i.

We now define a partition of $V(G)$ indexed by $V(L)$. Let $V_z \coloneqq \{(x, y) \in V(G) : x \in V(H), y \in S\}$. So $|V_z| = \operatorname{weight}(S) \leqslant m$. For each vertex x_i of $L - z$, let $V_{x_i} \coloneqq \{(x, y) \in V(G) : y \in V(J_i)\}$. So $|V_{x_i}| \leqslant |V(J_i)| \leqslant m$. Let $\mathcal{P} \coloneqq \{V_z\} \cup \{V_{x_i} : x \in V(L), i \in \{1, \ldots, t\}\}$. By construction, $\mathcal{P}$ is a partition of $V(G)$, and each part of $\mathcal{P}$ has size at most m.

We now verify that $\mathcal{P}$ is an L-partition of G. Consider an edge vv' of G. The goal is to show that vv' "maps" to a vertex or edge of L. If $v \in V_z$ or $v' \in V_z$ then vv' maps to a vertex or edge of L (since z is dominant in L). Otherwise, $v \in V_{x_i}$ and $v' \in V_{x'_j}$ for some $x, x' \in V(H)$. By the definition of V_{x_i}, we have $v = (x, y)$ for some $y \in V(J_i)$. Similarly, $v' = (x', y')$ for some $y' \in V(J_j)$. Since $vv' \in E(G)$, we have $x = x'$ or $xx' \in E(H)$, and $y = y'$ or $yy' \in E(J - S)$. If $yy' \in E(J - S)$, then y and y' are in the same component of $J - S$, implying $i = j$. If $y = y'$ then $i = j$ as well. In both cases, $x_i = x'_j$ or $x_i x'_j \in E(L)$. Hence, vv' maps to a vertex or edge of L.

This shows that $\mathcal{P}$ is an L-partition of G with width at most m. By Observation 1.2, G is contained in $L \boxtimes K_m$. □

Note that Lemma 6.5 holds even when J is only (n,m)-separable with integer-valued weights. Our goal now is to find graphs that are (n,m)-separable where $m \in O(n^{1-\epsilon})$.

Lemma 6.6. — *For every path P and $c,n \in \mathbb{N}$, the graph $G := P \boxtimes K_c$ is $(n,\sqrt{cn})$-separable.*

Proof. — Say $P = (v_1, v_2, \dots)$ and $V(K_c) = \{1,\dots,c\}$. Assume the vertices of G are assigned non-negative weights, with total weight n. Let $m := \lceil \sqrt{n/c} \rceil$. For $i \in \{1,\dots,m\}$, let $S_i := \{(v_j,\ell) : j \equiv i \pmod m, \ell \in \{1,\dots,c\}\}$. So $S_1,\dots,S_m$ is partition of $V(G)$. Thus $S_{i^\star}$ has total weight at most $\frac{n}{m}$ for some $i^\star \in \{1,\dots,m\}$. Each component of $G - S_{i^\star}$ has at most $c(m-1)$ vertices. The result follows since $\frac{n}{m} \leqslant \sqrt{cn}$ and $c(m-1) < \sqrt{cn}$. □

Lemmata 6.5 and 6.6 together imply:

Lemma 6.7. — *For any graph H, path P and $c \in \mathbb{N}$, if G is any n-vertex graph contained in $H \boxtimes P \boxtimes K_c$, then G is contained in $L \boxtimes K_m$ for some graph L with $\operatorname{tw}(L) \leqslant \operatorname{tw}(H)+1$, where $m \leqslant \sqrt{cn}$.*

Several recent results show that certain graphs G are contained in $H \boxtimes P \boxtimes K_c$, for some path P and graph H with bounded treewidth [30, 38, 41, 57, 58, 79]. In all these cases, Lemma 6.7 is applicable, implying that G is contained in $L \boxtimes K_{O(\sqrt{n})}$, for some graph L with bounded tree-width.

We give one example: map graphs. Start with a graph G embedded without crossings in a surface of Euler genus g, with each face labelled a "nation" or a "lake", where each vertex of G is incident with at most d nations. Let M be the graph whose vertices are the nations of G, where two vertices are adjacent in G if the corresponding faces in G share a vertex. Then M is called a *(g,d)-map graph*. Since the graphs of Euler genus g are precisely the $(g,3)$-map graphs [35], map graphs are a natural generalisation of graphs embeddable in surfaces. Distel, Hickingbotham, Huynh and Wood [30] proved that any (g,d)-map graph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$. The next result thus follows from Lemma 6.7.

Proposition 6.8. — *Every n-vertex (g,d)-map graph G is contained in $H \boxtimes K_m$ for some apex graph H with $\operatorname{tw}(H) \leqslant 4$, where $m \leqslant \sqrt{\ell n}$ and $\ell := \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$.*

Dujmović, Eppstein and Wood [35] showed that n-vertex (g,d)-map graphs have separation-number $\Theta(\sqrt{(g+1)(d+1)n})$. Thus Proposition 6.8 gives another example of a graph class with separation-number $cn^{1-\epsilon}$, where $\mathrm{tw}(H)$ is independent of c in the corresponding product structure theorem. (Compare with the discussion after Open Problem 1.6.)

Distel, Hickingbotham, Seweryn and Wood [31] showed that every (g,k)-planar graph G is contained in $H \boxtimes P \boxtimes K_{c(g,k)}$ for some graph H with $\mathrm{tw}(H) \leqslant 963\,922\,179$. Thus Lemma 6.7 implies:

Corollary 6.9. — *Every n-vertex (g,k)-planar graph G is contained in $L \boxtimes K_m$ for some graph L with $\mathrm{tw}(L) \leqslant 963\,922\,180$, where $m \leqslant \sqrt{c(g,k)n}$.*

Motivated by Lemma 6.5, we give three more examples of (n,m)-separable graphs. The first is a multi-dimensional generalisation of Lemma 6.6.

Proposition 6.10. — *For all paths $P_1,\dots,P_d$ and $c,n \in \mathbb{N}$, the graph $G \coloneqq P_1 \boxtimes \cdots \boxtimes P_d \boxtimes K_c$ is $(n,(dn)^{d/(d+1)}c^{1/(d+1)})$-separable.*

Proof. — Say $P_i = (1,2,\dots)$ and $V(K_c) = \{1,\dots,c\}$. Assume the vertices of G are assigned non-negative weights, with total weight n. Let $m \coloneqq \lceil (dn/c)^{1/(d+1)} \rceil$. For $j \in \{1,\dots,m\}$, let S_j be the set of all vertices $(x_1,\dots,x_d,\ell)$ in G where $x_i \equiv j \pmod{m}$ for some $i \in \{1,\dots,d\}$. Each vertex of G is in at least one and in at most d such sets. Thus, the total weight of $S_1,\dots,S_m$ is at most dn. Thus $S_{j^\star}$ has total weight at most $\frac{dn}{m}$ for some $j^\star \in \{1,\dots,m\}$. Each component of $G - S_{j^\star}$ has at most $c(m-1)^d$ vertices. The result follows since $\frac{dn}{m} \leqslant dn/(dn/c)^{1/(d+1)} = (dn)^{d/(d+1)}c^{1/(d+1)}$ and $c(m-1)^d \leqslant c(dn/c)^{d/(d+1)} = (dn)^{d/(d+1)}c^{1/(d+1)}$ □

Now consider trees.

Proposition 6.11. — *Every n-vertex tree T with maximum degree $\Delta \geqslant 3$ is $\left(n, \frac{(1+o(1))n}{\log_{\Delta-1} n}\right)$-separable.*

Proof. — Assume the vertices of T are assigned non-negative weights, with total weight n. In this proof all logs are base $\Delta - 1$. Let $m \coloneqq \lceil \log n - \log\log n \rceil \geqslant 2$. Root T at a leaf vertex r. So each vertex has at most $\Delta - 1$ children. For $j \in \{1,\dots,m\}$, let $V_j \coloneqq \{v \in V(T) : \mathrm{dist}_T(v,r) \equiv j \pmod{m}\}$. Thus $V_1,\dots,V_m$ is a partition of $V(T)$. There exists $j^\star$ such that $V_{j^\star}$ has weight at most $\frac{n}{m}$. In $T - V_{j^\star}$, each component has radius at most $m-2$, so the number of vertices is at most $\sum_{i=0}^{m-2}(\Delta-1)^i = ((\Delta-1)^{m-1}-1)/(\Delta-2) < (\Delta-1)^{m-1}$. The result follows since $\frac{n}{m} \leqslant \frac{n}{\log n - \log\log n} \leqslant \frac{(1+o(1))n}{\log n}$ and $(\Delta-1)^{m-1} < (\Delta-1)^{\log n - \log\log n} = \frac{n}{\log n}$. □

To get the $(n,o(n))$-separable result in Proposition 6.11, the bounded degree assumption is necessary. Suppose that for all n every star T is (n,m)-separable where $m < \frac{n}{3}$. Let T be the star with p leaves. Let r be the centre vertex of T. Assign each leaf a weight of 1 and assign r a weight of $\frac{p}{2}$. The total weight is $n := \frac{3p}{2}$. So for some $m < \frac{n}{3} = \frac{p}{2}$ there is a set S of vertices in T with weight at most m such that each component of $T - S$ has at most m vertices. Since r has weight $\frac{p}{2} > m$, we have $r \notin S$. So every vertex in S is a leaf, each of which has weight 1. So $|S| \leqslant m$. Thus $T - S$ is a star with at least $p - m > m$ leaves, which is a contradiction. Hence $m \geqslant \frac{n}{3}$.

Proposition 6.11 generalises as follows.

PROPOSITION 6.12. — *For all $\Delta, k \in \mathbb{N}$ there exists $\alpha > 0$ such that every graph G with maximum degree Δ and treewidth k is $\left(n, \frac{\alpha n}{\log n}\right)$-separable.*

Proof. — Let $c := 18(k+1)\Delta$. Distel and Wood [32] proved that G is contained in $T \boxtimes K_c$ for some tree T with maximum degree at most 6Δ. Observe that if H is any (n,m)-separable graph and $c \in \mathbb{N}$, then $H \boxtimes K_c$ is (n, cm)-separable. By Proposition 6.11, $T \boxtimes K_c$ is $\left(n, \frac{c(1+o(1))n}{\log_{6\Delta-1} n}\right)$-separable. Observe that if H is any (n,m)-separable graph, then every subgraph of H is (n,m)-separable. Thus G is $\left(n, \frac{c(1+o(1))n}{\log_{6\Delta-1} n}\right)$-separable. □

6.3. Bad News

We now present a result that highlights the difficulty of Open Problem 1.6. For simplicity, we focus on the $\epsilon = \frac{1}{2}$ case. Proposition 6.13 below shows there are n-vertex graphs G with $\text{td}(G) \leqslant \sqrt{n}$ such that G is contained in no graph $H \boxtimes K_m$ with $m \in O(\sqrt{n})$ and $\text{tw}(H)$ bounded.

PROPOSITION 6.13. — *For any $c \in \mathbb{N}$ there exist infinitely many $n \in \mathbb{N}$ for which there is an n-vertex graph G with $\text{td}(G) \leqslant \sqrt{n}$ such that for any graph H, if G is contained in $H \boxtimes K_m$ with $m \leqslant c\sqrt{n}$, then $\omega(H) \geqslant \frac{\log n}{4\log(c \log n)}$.*

Proof. — Fix any integer $\ell \geqslant 2$. Let $d := c\ell^2$ and $h := c^{\ell-1}\ell^{2\ell-4}$. Note that $h, \ell \in \mathbb{N}$.

For $j \in \{1, \dots, \ell\}$, let T_j be the complete d-ary tree of vertex-height j, and let T'_j be the $(h-1)$-subdivision of T_j. Consider $T'_1 \subseteq T'_2 \subseteq \cdots \subseteq T'_\ell$ with a common root vertex r.

CLAIM. — *For every partition $\mathcal{P}$ of $V(T'_j)$ with parts of size at most $\frac{hd-1}{\ell-1}$ there exists a root–leaf path in T'_j intersecting at least j parts of $\mathcal{P}$.*

Proof. — We proceed by induction on j. Since $|V(T'_1)| = 1$ the $j = 1$ case is trivial. Assume the $j-1$ case holds. Let $\mathcal{P}$ be any partition of $V(T'_j)$ with parts of size at most $\frac{hd-1}{\ell-1}$. By induction, there is a leaf v of T'_{j-1} such that the vr-path in T'_{j-1} intersects at least $j-1$ parts $P_1, \dots, P_{j-1}$ of $\mathcal{P}$. Let X be the set of descendants of v. If $X \subseteq P_1 \cup \dots \cup P_{j-1}$ then

$$hd = |X| \leqslant |P_1 \cup \dots \cup P_{j-1}| \leqslant (j-1)\, \tfrac{hd-1}{\ell-1} \leqslant hd - 1,$$

which is a contradiction. Thus there is a vertex $x \in X \setminus (P_1 \cup \dots \cup P_{j-1})$. Let y be any leaf-descendent of x. Thus, the yr-path in T'_j intersects at least j parts of $\mathcal{P}$, as desired. □

Let G be the closure of T'_ℓ. Let

$$n \coloneqq |V(G)| = h(\tfrac{d}{d-1})(d^{\ell-1} - 1) + 1.$$

We need the following lower bound on n:

$$n \geqslant hd^{\ell-1} = c^{\ell-1}\ell^{2\ell-4}(c\ell^2)^{\ell-1} = c^{2\ell-2}\ell^{4\ell-6}. \tag{6.1}$$

And we need the following upper bound on n:

$$\begin{aligned} n(\ell-1)^2 &= (h(\tfrac{d}{d-1})(d^{\ell-1} - 1) + 1)(\ell-1)^2 \\ &\leqslant h(\tfrac{d}{d-1})(d^{\ell-1})(\ell-1)^2 \\ &< hd^{\ell-1}\ell^2 \\ &= (c^{\ell-1}\ell^{2\ell-4})d^2(c\ell^2)^{\ell-3}\ell^2 \\ &= c^{2\ell-4}d^2\ell^{4\ell-8}. \end{aligned} \tag{6.2}$$

The vertex-height of T'_ℓ equals $h(\ell-1)+1$. By (6.1),

$$\operatorname{td}(G) = h(\ell-1)+1 \leqslant h\ell = c^{\ell-1}\ell^{2\ell-3} \leqslant \sqrt{n}.$$

Assume that G is contained in $H \boxtimes K_m$ for some graph H and integer $m \leqslant c\sqrt{n}$. By (6.2),

$$m(\ell-1) \leqslant c\sqrt{n}(\ell-1) < c^{\ell-1}\ell^{2\ell-4}d = hd.$$

Thus $m \leqslant \frac{hd-1}{\ell-1}$. By Observation 1.2, there is an H-partition of G with width at most $\frac{hd-1}{\ell-1}$. By the claim, there exists a root–leaf path in T'_ℓ intersecting at least ℓ parts of $\mathcal{P}$. These ℓ parts form a clique in H. Thus $\omega(H) \geqslant \ell$. By Eq. (6.2), $n < c^{2\ell}\ell^{4\ell}$, which implies $\omega(H) \geqslant \ell \geqslant \frac{\log n}{4\log(c\log n)}$, as desired. □

Proposition 6.13 is not a negative answer to Open Problem 1.6 since G is a single graph, not a hereditary class. Indeed, the graphs in Proposition 6.13 have unbounded complete subgraphs, and therefore are in no hereditary class with strongly sublinear separation-number. This result can be interpreted as follows: A natural strengthening of Open Problem 1.6 (with $\epsilon = \frac{1}{2}$) says that every n-vertex graph G with $\mathrm{td}(G) \in O(\sqrt{n})$ is contained in $H \boxtimes K_m$, for some graph H with $O(1)$ treewidth, where $m \in O(\sqrt{n})$. Proposition 6.13 says this strengthening is false. So it is essential that $\mathcal{G}$ is a hereditary class in Open Problem 1.6.

6.4. Future Directions

We conclude by listing several unsolved special cases of Open Problem 1.6 of particular interest:

- Does Open Problem 1.6 hold for touching graphs of 3-D spheres, which have $O(n^{2/3})$ separation-number [69]?
- Eppstein and Gupta [50] defined a graph G to be *k-crossing-degenerate* if G has a drawing in the plane such that the associated crossing graph is k-degenerate. They showed that such graphs have $O(k^{3/4}n^{1/2})$ separation-number. It is open whether Open Problem 1.6 holds for k-crossing-degenerate graphs. The same question applies for k-gap-planar graphs [4].
- Does Open Problem 1.6 hold for string graphs on m edges, which have $O(m^{1/2})$ separation-number [63, 64]?
- Does Open Problem 1.6 hold for graphs with layered tree-width k, which have $O(\sqrt{kn})$ separation-number [40]?
- See [45, 77] for many geometric intersection graphs where Open Problem 1.6 is unsolved and interesting.

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