Innov. Graph Theory **2**, 2025, pp. 275–299
https://doi.org/10.5802/igt.13

INNOVATIONS IN GRAPH THEORY

COLOURING THE 1-SKELETON OF d-DIMENSIONAL TRIANGULATIONS

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Abstract. — While every plane triangulation is colourable with three or four colours, Heawood showed that a plane triangulation is 3-colourable if and only if every vertex has even degree. In $d \geqslant 3$ dimensions, however, every $k \geqslant d+1$ may occur as the chromatic number of some triangulation of $\mathbb{S}^d$. As a first step, Joswig [14] structurally characterised which combinatorial triangulations of $\mathbb{S}^d$ have a $(d+1)$-colourable 1-skeleton. In the 20 years since Joswig's result, no characterisations have been found for any $k > d+1$.

In this paper, we structurally characterise which combinatorial triangulations of $\mathbb{S}^d$ have a $(d+2)$-colourable 1-skeleton: they are precisely the combinatorial triangulations that have a subdivision such that for every $(d-2)$-cell, the number of incident $(d-1)$-cells is divisible by three.

1. Introduction

We consider the problem of colouring the vertices of the 1-skeleton of triangulations of the d-dimensional sphere $\mathbb{S}^d$ for $d \geqslant 2$. In this paper, all triangulations are assumed to be combinatorial[(1)]. For each triangulation of $\mathbb{S}^d$, we need at least $d+1$ colours since every d-cell induces a complete graph on $d+1$ vertices in the 1-skeleton.

For $d = 2$, Heawood [12] showed that a plane triangulation is colourable with three colours if and only if every vertex has even degree. See also [10, 15, 18, 22] for alternative proofs and variations of Heawood's theorem. On the other hand, by the four-colour theorem [1], *every* plane triangulation is colourable with four colours.

Keywords: colouring, triangulation, simplicial complex, Heawood's theorem, four colour theorem.
2020 *Mathematics Subject Classification:* 05C15, 05E45, 05C10.
(*) Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 546892829.
(1) We define 1-skeleton and triangulation in Section 2. For the definition of *combinatorial* see e.g. [14] or [16].

For $d \geqslant 3$, however, there exists for every $k \geqslant 1$ a triangulation of $\mathbb{S}^d$ whose 1-skeleton is the complete graph K_{d+k} ([23, Example 0.6] and Example 5.4). Naturally, this raises the following question:

QUESTION 1.1. — *Let $k \geqslant 1$ be an integer. Can you find a structural characterisation for all $d \geqslant 3$ of the triangulations of $\mathbb{S}^d$ whose 1-skeleton is $(d+k)$-colourable?*

Joswig [14] answered Question 1.1 for $k = 1$ as follows:

THEOREM 1.2 (Joswig [14], "moreover"-part by Carmesin, Nevinson and Saunders [6]). — *Let $d \geqslant 2$ be an integer and let C be a triangulation of $\mathbb{S}^d$. Then the following assertions are equivalent.*

(1) The 1-skeleton of C has a proper $(d+1)$-colouring.
(2) All $(d-2)$-cells of C are incident with an even number of $(d-1)$-cells.

Moreover, if $d = 3$, then we may add:

(3) There exists a 3-edge-colouring of C such that every 2-cell contains edges of all colours.

In the 20 years since Joswig's result, no characterisations have been found for any $k > 1$. Heawood [12] observed that the four-colour theorem is equivalent to the statement that every plane triangulation G has a subdivision(2) G' such that all vertices in G' have degree divisible by three, see Figure 1.1. Carmesin [3] asked for a characterisation of the triangulations of $\mathbb{S}^3$ that admit subdivisions(3) such that the number of 2-cells incident with each edge is divisible by three. We answer his question, even more generally in arbitrary dimensions, and thereby also solve Question 1.1 for $k = 2$.

THEOREM A. — *Let $d \geqslant 2$ be an integer and let C be a triangulation of $\mathbb{S}^d$. Then the following assertions are equivalent.*

(1) The 1-skeleton of C has a proper $(d+2)$-colouring.
(2) There exists a subdivision C' of C such that for every $(d-2)$-cell, the number of incident $(d-1)$-cells is divisible by three.

Moreover, if $d = 3$, then we may add:

(3) The maximal subdivision of C has a 2-edge-colouring such that every 2-cell contains edges of both colours.

(2) A plane triangulation G' is a *subdivision* of a plane triangulation G if G' is obtained from G by adding a vertex v_f in some faces f of G and joining each v_f to all vertices in the boundary of f.

(3) For the definition of a *subdivision* of a triangulation of $\mathbb{S}^d$ for $d \geqslant 3$ see Section 5.

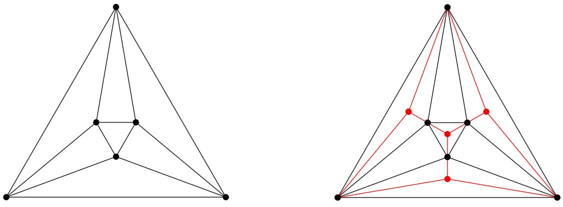

Figure 1.1. (a) A plane triangulation G and (b) a subdivision G' of G such that all vertices have degree divisible by three. The added vertices and edges are red.

In contrast to Joswig's result, that allows to construct a proper $(d+1)$-colouring of the 1-skeleton of any given triangulation of $\mathbb{S}^d$ in polynomial time, we show that deciding whether the 1-skeleton of a given triangulation of $\mathbb{S}^d$ is $(d+2)$-colourable is NP-complete for each $d \geqslant 3$.

To prove Theorem A, I independently found a stronger version of Joswig's method, in the form of the Local-Global Colouring Lemma, see Lemma 1.3 below. Indeed, we use the Local-Global Colouring Lemma to prove both Theorem A and Theorem 1.2.

Roughly, the idea of the Local-Global Colouring Lemma is as follows. Let C be a triangulation of $\mathbb{S}^d$, whose vertices we want to colour, and let s_0 be an arbitrary d-cell of C. We start by assigning distinct colours to the vertices on the boundary of s_0. Now suppose that for every two d-cells s and t of C sharing a $(d-1)$-cell we are given a map $g_{\vec{st}}$ that determines the colours of the vertices on the boundary of t, given any colouring of the vertices on the boundary of s. Then we greedily extend the colouring of the vertices on the boundary of s_0 to a colouring c of C via the maps $g_{\vec{st}}$.

The Local-Global Colouring Lemma states that the colouring c is well-defined and proper if the maps $g_{\vec{st}}$ are compatible in the following sense: On the one hand, each map $g_{\vec{st}}$ must fix the colours of the vertices of the $(d-1)$-cell shared by s and t. On the other hand, cyclically going around a $(d-2)$-cell with the maps $g_{\vec{st}}$ yields the identity. A collection of maps $g_{\vec{st}}$ that satisfy both conditions is called a *proper canonical local colouring*, see Section 3.

Lemma 1.3 (Local-Global Colouring Lemma). — *Let $d \geqslant 2$ be an integer. A triangulation of $\mathbb{S}^d$ is canonically locally k-colourable if and only if its 1-skeleton is k-colourable.*

1.1. Related work

There exists a rich literature on extensions of theorems from two dimensions to three dimensions. For example, Carmesin [4] proved a 3-dimensional analogue of Kuratowski's theorem. Carmesin and Mihaylov [5] extended the concept and excluded-minors characterisation of outerplanar graphs. Kurkofka and Nevinson [17] asked for the least integer k such that every simplicial 2-complex embedded in $\mathbb{S}^3$ has a k-edge-colouring, and showed $k \leqslant 12$. Georgakopoulos and Kim [9] extended Whitney's Theorem on unique embeddings of 3-connected planar graphs.

1.2. Organization of the paper

In Section 2 we introduce the necessary definitions and terminology that will be used in this paper. In Section 3 we prove the Local-Global Colouring Lemma 1.3. In Section 4 we prove (1) $\Leftrightarrow$ (2) of Theorem 1.2. In Section 5 we prove (1) $\Leftrightarrow$ (2) of Theorem A and show that for each $d \geqslant 3$ it is NP-complete to decide whether the 1-skeleton of a given triangulation of $\mathbb{S}^d$ is $(d+2)$-colourable. In Section 6 we prove the "moreover"-part of Theorem A. In Section 7, we show how to obtain edge-colourings and face-colourings from vertex-colourings of triangulations of $\mathbb{S}^d$. Then, we conclude the paper with open problems in Section 8.

2. Definitions and Terminology

2.1. Topology

For the background in algebraic topology, we refer to Hatcher's book [11].

We assume familiarity with [7, Sections 2.4, 2.5 and 4.1]. In particular, let G be a graph and $\mathcal{C}$ be a set of cycles in G. If v is a vertex in G then $\pi_1^{\mathcal{C}}(G, v)$ is the subgroup of $\pi_1(G, v)$ generated by cycles in $\mathcal{C}$. Moreover, $\mathcal{C}$ *generates* a cycle c in G if there exists a vertex v in G such that some walk in $\pi_1^{\mathcal{C}}(G, v)$ stems from c.

A *triangulation of* $\mathbb{S}^d$ is a simplicial d-complex embedded in $\mathbb{S}^d$ such that the underlying space is $\mathbb{S}^d$, see [23, Example 5.2 (iii)]. In this paper we only consider triangulations of $\mathbb{S}^d$ with $d \geqslant 2$. Throughout this paper we assume that every triangulation of $\mathbb{S}^d$ comes with an ordering $<$ of its vertices. The d'*-skeleton* of a cell complex C is the cell complex C' consisting of all i-cells in C with $i \leqslant d'$. Given a cell complex C embedded in $\mathbb{S}^d$, a *chamber* of C is a connected component of $\mathbb{S}^d \setminus C$.

We say that a vertex v is a vertex of a cell f if v is contained in f. Two d-cells s and t *share* the d'-cell f if f is included in both s and t. If a d-cell t contains vertices $v_1, v_2, \dots, v_k$, we say that t is a d-cell *on the vertices* $v_1, v_2, \dots, v_k$.

For the definition of the *dual cell complex* C^* of a triangulation C of $\mathbb{S}^d$ see [2] or [20, Section 64]. In particular, C^* is again embedded in $\mathbb{S}^d$ and every d'-cell of C (for $0 \leqslant d' \leqslant d$) corresponds to a $(d-d')$-cell of C^*. Given a triangulation C of $\mathbb{S}^d$, the *dual 2-complex* of C is the 2-skeleton of the dual cell complex of C, and the *dual graph* of C is the 1-skeleton of the dual cell complex of C. Note that the dual graph of C may have parallel edges.

Observe that every $(d-1)$-cell in a triangulation C of $\mathbb{S}^d$ is included in exactly two d-cells. Let f be a $(d-2)$-cell in C and $c = t_0 f_1 t_1 f_2 t_2 \dots t_{k-1} f_k t_k$ be the cyclic ordering of the $(d-1)$-cells $f_1, \dots, f_k$ and d-cells $t_0, t_1, \dots,$ $t_k = t_0$ around f induced by the embedding of the triangulation C in $\mathbb{S}^d$. Then c is a cycle in the dual graph G of C, which we call the *f-cycle in G*.

The dual graph G of a triangulation C of $\mathbb{S}^d$ can also be constructed as follows. The set $V(G)$ of vertices is the set of d-cells of C. For each $(d-1)$-cell that is included in two d-cells s and t of C we add an edge st to G. The dual 2-complex D of C can be obtained from the dual graph G of C by adding all f-cycles (for all $(d-2)$-cells f in C) as 2-cells to D.

It is well-known that the d-dimensional sphere $\mathbb{S}^d$ is orientable. We assume throughout this paper that we are given $\mathbb{S}^d$ together with an orientation. Let C be a triangulation of $\mathbb{S}^d$ and t be a d-cell of C. The orientation of $\mathbb{S}^d$ induces an orientation on every d-simplex embedded in $\mathbb{S}^d$ and therefore on the d-cell t. This orientation of t determines a vertex ordering $v_0, v_1, \dots, v_d$ of the vertices in the boundary of t up to a permutation of even parity. We say that such a vertex ordering determined by the orientation of $\mathbb{S}^d$ is *positive*, and all other vertex orderings of t are called *negative*. See [11, 20] for more details.

2.2. Gain graphs

Throughout this paper we assume that all directed graphs have no loops. For a directed graph G and a pair $\vec{e} = uv$ of vertices, we define $\overleftarrow{e} = vu$. A directed graph G is *symmetric* if for every pair $\vec{e} = uv$ of vertices, $\vec{e} \in E(G)$ if and only if $\overleftarrow{e} \in E(G)$. Let $(\Gamma, \cdot)$ be a finite group with neutral element $\mathbb{1}$.

A *gain graph* is a symmetric directed graph G together with an assignment of weights $a_{\vec{e}} \in \Gamma$ to every directed edge $\vec{e}$ such that $a_{\overleftarrow{e}} = a_{\vec{e}}^{-1}$.

Let G be a gain graph and $W = v_0\vec{e}_1v_1\vec{e}_2v_2 \dots v_{k-1}\vec{e}_kv_k$ be a walk in G with $\vec{e}_i = v_{i-1}v_i$ all i. The *gain value* ℓ_W of the walk W is defined by $\ell_W = a_{\vec{e}_k} \dots a_{\vec{e}_2}a_{\vec{e}_1} \in \Gamma$. Note that, in general, the gain value of a closed walk depends on the start point and the direction of the closed walk. A closed walk c is *balanced* if $\ell_c = \mathbb{1}$, which is independent of the start point and direction of the closed walk.

3. The Local-Global Colouring Lemma

In this section, we prove the Local-Global Colouring Lemma 1.3.

Fact 3.1. — *Let Γ be a group. Let G be a gain graph with weights in Γ and $\mathcal{C}$ be a set of cycles of G that generate all cycles of G. If every cycle in $\mathcal{C}$ is balanced, then every cycle in G is balanced.* □

Lemma 3.2. — *Let G be the dual graph of a triangulation C of $\mathbb{S}^d$. Then the set of all f-cycles in G (for $(d-2)$-cells f in C) generates all cycles in G.*

Proof. — Let C^* be the dual cell complex of C and recall that the 2-skeleton of C^* is the dual 2-complex D of C. Observe that C^* is simply connected. Then it follows that the 2-skeleton of C^*, i.e. the dual 2-complex D, is again simply connected, see Hatcher [11, Proposition 1.26 (b)]. By construction, the face cycles (i.e. the boundaries of the 2-cells) of D are exactly the cycles in $\mathcal{C}$. Since D is simply connected, the face cycles of D generate all cycles in the 1-skeleton of D (see [11, Proposition 1.26 (a)]) and therefore in G. □

The following corollary follows immediately from Fact 3.1 together with Lemma 3.2.

Corollary 3.3. — *Let G be a dual gain graph of a triangulation of $\mathbb{S}^d$. If every f-cycle in G is balanced, then every cycle in G is balanced.* □

For a finite set X, we define an *X-gain graph* to be a gain graph where the weights on the edges are permutations of X, i.e. for every directed edge $\vec{e} \in E(G)$ there exists a bijective map $g_{\vec{e}}\colon X \to X$ such that $g_{\overleftarrow{e}} = g_{\vec{e}}^{-1}$. Again, we say that an X-gain graph is a *dual X-gain graph* of a triangulation C of $\mathbb{S}^d$ if the corresponding undirected graph is the dual graph of C.

Let G be an X-gain graph with bijective maps $g_{\vec{e}}\colon X \to X$. We say that a colouring $\phi\colon V(G) \to X$ of the vertices of G with elements of X *commutes* with $g_{\vec{e}}$ for some edge $\vec{e} = uv$ in G if $g_{\vec{e}}(\phi(u)) = \phi(v)$ holds. Observe that if ϕ commutes with $g_{\vec{e}}$, it also commutes with $g_{\overleftarrow{e}}$ since $g_{\overleftarrow{e}}(\phi(v)) = g_{\vec{e}}^{-1}(\phi(v)) = \phi(u)$.

Lemma 3.4. — *Let G be an X-gain graph with maps $g_{\vec{e}}$ assigned to the directed edges $\vec{e}$. Let $\mathcal{C}$ be a set of cycles of G that generate all cycles in G. If every cycle in $\mathcal{C}$ is balanced, then there exists a colouring $\phi\colon V(G) \to X$ of G with elements of X that commutes with all maps $g_{\vec{e}}$.*

Proof. — Since the cycles in $\mathcal{C}$ generate all cycles in G, we have that all cycles in G are balanced by Fact 3.1. Let $u \in V(G)$ be an arbitrary vertex in G to which we assign an arbitrary colour $\phi(u) \in X$. Let T be a spanning tree of G. For each vertex v in G, let $P_v = u\vec{e}_1 v_1 \dots v_{k-1}\vec{e}_k v$ be the path in T from u to v. We define the colour of v to be $\phi(v) = (g_{\vec{e}_k} \circ \cdots \circ g_{\vec{e}_1})(\phi(u))$. Clearly, this colouring commutes with all maps $g_{\vec{e}}$ where $\vec{e}$ is an edge of the spanning tree T. For an edge $\vec{e'} = w_1 w_2 \in E(G) \setminus E(T)$, let P be the path from w_1 to w_2 in T. Since the cycle $c = w_1 P w_2 \overleftarrow{e'} w_1$ is balanced (i.e. has gain value $g_c = \mathbb{1}$) and ϕ commutes with all $g_{\vec{e}}$ for edges $\vec{e}$ in T, it also commutes with $g_{\vec{e'}}$. Therefore, the colouring ϕ of the vertices of G commutes with all $g_{\vec{e}}$. $\square$

Now, we formally define what it means for a triangulation of $\mathbb{S}^d$ to be canonically locally k-colourable. Recall that we assume that every triangulation of $\mathbb{S}^d$ comes with an ordering $<$ of its vertices. We denote by Σ_k the symmetric group whose elements are the permutations on $\{0, 1, \dots, k-1\}$. Let C be a triangulation of $\mathbb{S}^d$ and let $k \geqslant d+1$ be an integer. Given a d-cell t of C and a permutation $\pi \in \Sigma_k$, we call (t, π) a *colouring* of t. If the d-cell t has vertices $u_0 < u_1 < \cdots < u_d$ in its boundary, the colour of u_i *induced* by (t, π) is $\pi(i)$. Given two d-cells t and t' incident to the same d'-cell f' and two permutations $\pi, \pi' \in \Sigma_k$, we say that the colourings (t, π) and (t', π') *agree* on f' if, for each vertex u of f', the colour of u induced by (t, π) is equal to the colour of u induced by (t', π).

Definition 3.5 (Canonically Locally Colourable). — *Let C be a triangulation of $\mathbb{S}^d$ and $k \geqslant d+1$. A* proper canonical local k-colouring *of C is a dual Σ_k-gain graph G of C with bijective maps $g_{\vec{e}}\colon \Sigma_k \to \Sigma_k$ on the directed edges of G such that the following two conditions hold.*

1. *For every $(d-2)$-cell f in C, the f-cycle c in G is balanced.*
2. *For every $(d-1)$-cell f' in C and corresponding edge $\vec{e} = st$ in G, each colouring $(s, \pi_s) \in \Sigma_k$ agrees with the colouring $(t, g_{\vec{e}}(\pi_s))$ on f'.*

We say that C is canonically locally k-colourable *if there exists a proper canonical local k-colouring of C.*

Proof of the Local-Global Colouring Lemma (Lemma 1.3).

($\Rightarrow$). — Let C be a triangulation of $\mathbb{S}^d$ that is canonically locally k-colourable. That is, there exists a dual Σ_k-gain graph G with bijective maps $g_{\vec{e}}\colon \Sigma_k \to \Sigma_k$ on the directed edges $\vec{e}$ that satisfy Definition 3.5 (1) and (2). We need to show that the 1-skeleton of C has a proper k-colouring.

By Lemma 3.2 and Lemma 3.4, there exists a colouring of the vertices of G with elements from Σ_k that commutes with all maps $g_{\vec{e}}$. We denote by $\pi_t \in \Sigma_k$ the colour assigned to the vertex t of G.

In the first step, we show that for each vertex u of C, the colourings (t, π_t) of all incident d-cells t agree on u. Let s and t be two d-cells incident to u. Then there exists a path $t_0 t_1 t_2 \dots t_{m-1} t_m$ in the dual graph G with $t_0 = s$ and $t_m = t$ such that the d-cells $t_0, t_1, \dots, t_k$ are all incident to u. By Definition 3.5 (2), the colourings (t_i, π_{t_i}) and $(t_{i+1}, \pi_{t_{i+1}})$ of two adjacent d-cells t_i and t_{i+1} agree on the colour of u. Hence, all colourings (t_i, π_{t_i}) agree on the colour of u. It follows that any two d-cells incident to u agree on the colour of u.

Then we define the colour of u to be the colour of u induced by the colourings of the incident d-cells. This is a colouring of the 1-skeleton with elements in $\{0, 1, \dots, k-1\}$. To see that this colouring is proper, let $u_1 u_2$ be an edge in the 1-skeleton of C. Then there exists a d-cell s that is incident to both u_1 and u_2. Since s is coloured with an element $\pi_s \in \Sigma_k$, it assigns distinct colours to u_1 and u_2.

($\Leftarrow$). — Let $\psi\colon V \to \{0, 1, \dots, k-1\}$ be a proper k-colouring of the 1-skeleton of C. For each d-cell t of C, we fix a colouring (t, π_t) with $\pi_t \in \Sigma_k$ such that for each vertex v in the boundary of t, the colour of u induced by (t, π_t) is $\psi(u)$. To construct a proper canonical local $(k+1)$-colouring of C, we have to define a dual Σ_k-gain graph of C. For each edge $\vec{e} = st$ in the dual graph G and for each $\sigma_s \in \Sigma_k$, we define $g_{\vec{e}}(\sigma_s) = \sigma_s \circ \pi_s^{-1} \circ \pi_t$.

Obviously, $g_{\vec{e}}$ is a bijective map. Moreover, observe that if $\sigma_t = g_{\vec{e}}(\sigma_s)$ then $g_{\overleftarrow{e}}(\sigma_t) = \sigma_t \circ \pi_t^{-1} \circ \pi_s = \sigma_s$ since $\sigma_t \circ \pi_t^{-1} = \sigma_s \circ \pi_s^{-1}$. This proves $g_{\vec{e}} = g_{\overleftarrow{e}}^{-1}$ for all edges $\vec{e}$. It remains to check that Definition 3.5 (1) and (2) are satisfied.

(1). — Let f be a $(d-2)$-cell in C with corresponding f-cycle $c = t_0 \vec{e}_1 t_1 \dots t_{\ell-1} \vec{e}_{\ell-1} t_0$ in the dual graph G. Let $\sigma_{t_0} \in \Sigma_k$ be arbitrary. Then

$$\begin{aligned} g_c(\sigma_{t_0}) &= (g_{\vec{e}_{\ell-1}} \circ \dots \circ g_{\vec{e}_1})(\sigma_{t_0}) \\ &= \sigma_{t_0} \circ (\pi_{t_0}^{-1} \circ \pi_{t_1}) \circ (\pi_{t_1}^{-1} \circ \pi_{t_2}) \circ \dots \circ (\pi_{t_{\ell-1}}^{-1} \circ \pi_{t_0}) = \sigma_{t_0} \ . \end{aligned}$$

(2). — Let s and t be two d-cells in C that share a $(d-1)$-cell f with corresponding edge $\vec{e} = st$ in the dual graph G. Let u be an arbitrary vertex of f with index i in the ordering of the vertices of s and with index j in the ordering of the vertices of t. Observe that $\pi_s(i) = \pi_t(j) = \psi(u)$. Let $\sigma_s \in \Sigma_k$ be arbitrary. By definition, the colouring (s, σ_s) of s induces the colour $\sigma_s(i)$ on u. Then,

$$(g_{\vec{e}}(\sigma_s))(j) = (\sigma_s \circ \pi_s^{-1} \circ \pi_t)(j) = \sigma_s(\pi_s^{-1}(\pi_t(j))) = \sigma_s(\pi_s^{-1}(\pi_s(i))) = \sigma_s(i),$$

which proves that (s, σ_s) and $(t, \sigma_t = g_{\vec{e}}(\pi'_s))$ induce the same colour on the vertex u. □

4. Proof of (1) ⇔ (2) of Theorem 1.2

For convenience, we include a proof of Joswig's Theorem 1.2. Readers familiar with [14] are encouraged to skip to Section 5.

Let C be a triangulation of $\mathbb{S}^d$ and recall that we assume that we are given $\mathbb{S}^d$ together with a fixed orientation. Recall that this induces a fixed orientation on every d-cell t of C. Moreover, recall that this defines positive and negative vertex orderings of the vertices in the boundary of t. We can use this to define the orientation of a d-cell with respect to a given $(d+1)$-colouring ψ as follows.

Definition 4.1 (Orientation of properly $(d+1)$-coloured d-cells). — *Let C be a triangulation of $\mathbb{S}^d$ with a proper $(d+1)$-colouring $\psi\colon V \to \{0,\dots,d\}$ of its 1-skeleton. For a d-cell t of C, let $v_0,\dots,v_d$ be a positive vertex-ordering. Then the ψ-orientation of t is* positive *if $i \mapsto \psi(i)$ is an even permutation, and otherwise* negative.

Observe that for a proper $(d+1)$-colouring ψ, a d-cell t has positive ψ-orientation if and only if $\psi^{-1}(0), \psi^{-1}(1), \dots, \psi^{-1}(d)$ is a positive vertex-ordering. Moreover, note that whether the ψ-orientation of a d-cell t is positive/negative does *not* depend on the chosen vertex-ordering $v_0,\dots,v_d$ of t, as long as the vertex-ordering is positive. In particular, if $v_0,\dots,v_d$ is a negative vertex-ordering, then the parity of the permutation

$$\begin{pmatrix} 0 & 1 & \dots & d \\ \psi(v_0) & \psi(v_1) & \dots & \psi(v_d) \end{pmatrix}$$

is even if and only if the ψ-orientation of t is negative. Analogously, choosing a reverse orientation of $\mathbb{S}^d$ flips all ψ-orientations of d-cells.

In order to prove (1) ⇔ (2) of Theorem 1.2, we show the following strengthening.

Theorem 4.2 (Joswig [14]). — *Let C be a triangulation of $\mathbb{S}^d$. Then the following statements are equivalent.*

(1) The 1-skeleton of C is $(d+1)$-colourable.
(2) There exists a proper $(d+1)$-colouring ψ of the 1-skeleton of C such that for every d-cell t of C, every d-cell that shares a $(d-1)$-cell with t in C has the reverse ψ-orientation to t.
(3) The dual graph of C is bipartite.
(4) All $(d-2)$-cells of C are incident with an even number of $(d-1)$-cells.

Proof.

(1) ⇒ (2). — Let C be a triangulation of $\mathbb{S}^d$ with a proper $(d+1)$-colouring $\psi\colon V \to \{0,1,\dots,d\}$ of its 1-skeleton. We show that two d-cells s and t that share a $(d-1)$-cell f have opposite ψ-orientations. Indeed, let u be the vertex of s not in f, and let v be the vertex of t not in f. Then it must hold that $\psi(u) = \psi(v)$. Let $u_0,\dots,u_{d-1}$ be the vertices of f. Then the vertex-orderings $u_0,\dots,u_{d-1},u$ of s, and $u_0,\dots,u_{d-1},v$ of t induce opposite orientations, but the vertex-colours are ordered in the same way. Therefore, s and t have different ψ-orientations.

(2) ⇒ (3). — Let G be the dual graph of C. Let $V_+ \subseteq V(G)$ be the set of all d-cells with positive ψ-orientation and let $V_- \subseteq V(G)$ be the set of all d-cells with negative ψ-orientation. Observe that this defines a bipartition $V(G) = V_+ \dot\cup V_-$. By (2), no two d-cells $s,t \in V(G)$ in the same part of this bipartition are adjacent in G.

(3) ⇒ (4). — Let f be a $(d-2)$-cell in C and let G be the dual graph of C. Let $c = t_0 f_1 t_1 f_2 t_2 \dots f_{\ell-1} t_{\ell-1} f_\ell t_0$ be the cyclic ordering of the $(d-1)$-cells $f_1,\dots,f_\ell$ and d-cells $t_0,\dots,t_{\ell-1}$ around f, which describes a cycle in G (see Section 2.1). By (3), G is bipartite and hence ℓ is even, showing that f is incident with an even number of $(d-1)$-cells in C.

(4) ⇒ (1). — Let C be a triangulation of $\mathbb{S}^d$ and assume that all its $(d-2)$-cells are incident with an even number of $(d-1)$-cells. Let $X = \Sigma_{d+1}$ be the symmetric group whose elements are the permutations on $\{0,1,2,\dots,d\}$. Recall that for a d-cell t with vertices $u_0 < u_1 < \dots < u_d$, each element $\pi_t \in X$ corresponds to a colouring of the vertices of t with $d+1$ colours where vertex u_i gets colour $\pi_t(i)$.

We will use the Local-Global Colouring Lemma (Lemma 1.3) to show the implication (4) ⇒ (1). For this, we construct a proper canonical local $(d+1)$-colouring of C, as defined in Definition 3.5. First, we need to construct the bijective maps $g_{\vec{e}}\colon X \to X$ on the directed edges $\vec{e}$ of the dual graph G. Then it suffices to show that these bijective maps satisfy Definition 3.5 (1) and (2). By Lemma 1.3, we then get a proper $(d+1)$-colouring of the 1-skeleton of C.

Let G be the dual graph of C. We assign bijective maps $g_{\vec{e}}\colon X \to X$ to the oriented edges $\vec{e}$ of G, which makes it into a dual X-gain graph of C, as follows. Let f be the $(d-1)$-cell in C corresponding to the edge $\vec{e} = st$ of G. For each colouring (s, π_s) of s (with $\pi_s \in X$) there exists a unique colouring (t, π_t) of t (with $\pi_t \in X$) that agrees with (s, π_s) on the vertices of f, and vice versa. We define $g_{\vec{e}}$ to be the bijection that maps each colouring $\pi_s \in X$ to the according π_t. Observe that $g_{\overleftarrow{e}} = g_{\vec{e}}^{-1}$ for all directed edges $\vec{e} \in E(G)$.

We prove that Definition 3.5 1 is fulfilled in the following claim.

Claim 4.3. — *For each $(d-2)$-cell f in C, the f-cycle c in G is balanced.*

Proof of Claim. — Let $c = s_0\vec{e}_1 s_1 \vec{e}_2 s_2 \dots s_{k-1}\vec{e}_k s_0$ be the f-cycle in G and f_i be the $(d-1)$-cell in C corresponding to the edge $\vec{e}_i$ in G, for $i = 1, \dots, k$. We have to show that $g_c = g_{\vec{e}_k} \circ \dots \circ g_{\vec{e}_2} \circ g_{\vec{e}_1} = \mathbb{1}$.

By definition of $g_{\vec{e}_i}$, the colouring $(s_{i-1}, \pi_{s_{i-1}})$ and the colouring $(s_i, \pi_{s_i} = g_{\vec{e}_i}(\pi_{s_{i-1}}))$ in C agree on the vertices of f_i and therefore on the vertices of f. Let (s_0, π_{s_0}) be an arbitrary colouring of s_0. Define $\pi_{s_i} := (g_{\vec{e}_i} \circ \dots \circ g_{\vec{e}_1})(\pi_{s_0})$, which gives colourings (s_i, π_{s_i}) of s_i. Then all colourings (s_i, π_{s_i}) agree on the vertices of the $(d-2)$-cell f. Without loss of generality, f is coloured with the colours $\{2, 3, \dots, d\}$. For $i = 1, \dots, k$, let w_i be the vertex of f_i that is not a vertex of f. Then the w_i are coloured alternatingly with the colours 0 and 1. By (4), every f-cycle has even length (i.e., k is even) and it follows that $g_c(\pi_{s_0}) = \pi_{s_0}$ for every colouring π_{s_0} of the vertices of s_0. Therefore, $g_c = \mathbb{1}$ for every f-cycle c in G. □

Note that Definition 3.5 (1) is fulfilled by Claim 4.3, and Definition 3.5 (2) is fulfilled by the definition of the functions $g_{\vec{e}}$. We use Lemma 1.3 with $k = d + 1$ to obtain a $(d + 1)$-colouring of the 1-skeleton of C. □

5. Proof of (1) ⇔ (2) of Theorem A

Let $\iota\colon C \hookrightarrow \mathbb{S}^d$ witness that C is a triangulation of $\mathbb{S}^d$ and let t be a d-cell of C. The d-dimensional simplicial complex C' that is obtained from C by *subdividing* t is defined as follows. First, we remove t from C. Then, we add a vertex (i.e. a 0-cell) v, and for each $i = 1, 2, \dots, d$ and for each $(i-1)$-cell f contained in t, we add an i-cell f' consisting of v and the vertices of f. Observe that C' is again a triangulation of $\mathbb{S}^d$ (map v to some point in the interior of $\iota(t)$). For triangulations C and C' of $\mathbb{S}^d$, we say that C' is a *subdivision* of C if there exists a subset T of d-cells of C such that C' is obtained from C by subdividing every d-cell of T. We say that a triangulation C of $\mathbb{S}^d$ is *subdividable* if there exists a subdivision C' of C such that for each $(d-2)$-cell f in C', the number of incident $(d-1)$-cells of f in C' is divisible by three.

Definition 5.1 (Orientation of properly $(d+2)$-coloured d-cells). — *Let C be a triangulation of $\mathbb{S}^d$ with a proper $(d+2)$-colouring $\psi\colon V \to \{0,\dots,d+1\}$ of its 1-skeleton. For a d-cell t of C, let $v_0,\dots,v_d$ be a positive vertex-ordering. Then the ψ-orientation of t is* positive *if*

$$\begin{pmatrix} 0 & 1 & \cdots & d & d+1 \\ \psi(v_0) & \psi(v_1) & \cdots & \psi(v_d) & c \end{pmatrix} \quad \textit{is an even permutation,}$$

where c is the colour not used by $v_0,\dots,v_d$, i.e. $\{c\} = \{0,\dots,d+1\} \setminus \psi(\{v_0,\dots,v_d\})$, and otherwise negative.

In order to prove (1) ⇔ (2) of Theorem A, we show the following strengthening.

Theorem 5.2. — *Let C be a triangulation of $\mathbb{S}^d$. Then the following statements are equivalent.*

(1) The 1-skeleton of C is $(d+2)$-colourable.
(2) There exists a subdivision C' of C and a proper $(d+2)$-colouring ψ' of the 1-skeleton of C' such that all d-cells of C' have the same ψ'-orientation.
(3) There exists a subdivision of C such that for all $(d-2)$-cells, the number of incident $(d-1)$-cells is divisible by three.

Proof.

(1) ⇒ (2). — Let C be a triangulation of $\mathbb{S}^d$ with a proper $(d+2)$-colouring ψ. Let G be the dual graph of C and $V_+, V_- \subseteq V(G)$ be the set of all d-cells of C with positive (respectively negative) ψ-orientation. Observe that V_+ and V_- partition the set of all d-cells, i.e. $V(G) = V_+ \dot\cup V_-$. We subdivide all d-cells in V_- to obtain the triangulation C' of $\mathbb{S}^d$. Let ψ' be the proper $(d+2)$-colouring of C' with $\psi'(v) = \psi(v)$ for all vertices $v \in V(C)$, i.e. the (unique) natural extension of the colouring ψ to the triangulation C'. It remains to show that all d-cells have positive ψ'-orientation.

If t is a d-cell of C that has positive ψ-orientation, then t is still a d-cell of C' with a positive ψ'-orientation. So let t be a d-cell of C with negative

ψ-orientation. Let u be the vertex that is added in the inside of t when subdividing t. Let t' be a d-cell of C' contained in t. Then t' consists of the vertices of a $(d-1)$-cell f (say, with vertices $v_0 < v_1 < \cdots < v_{d-1}$) of t and the vertex u. Let v be the vertex of the d-cell t that is not a vertex of the d-cell t'. First of all, observe that the vertex-ordering $v_0 < v_1 < \cdots < v_{d-1} < v$ induces the same orientation on the d-cell t as the vertex-ordering $v_0 < v_1 < \cdots < v_{d-1} < u$ does on the d-cell t'. Moreover, note that $\psi'(u) \neq \psi'(v)$ since the vertices u and v are adjacent in the 1-skeleton of C'. Therefore, the permutations

$$\begin{pmatrix} 0 & \dots & d-1 & d & d+1 \\ \psi'(v_0) & \dots & \psi'(v_{d-1}) & \psi'(v) & \psi'(u) \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \dots & d-1 & d & d+1 \\ \psi'(v_0) & \dots & \psi'(v_{d-1}) & \psi'(u) & \psi'(v) \end{pmatrix}$$

have opposite parities. Since the ψ-orientation of the d-cell t is negative, the ψ'-orientation of the d-cell t' is positive. This shows that all d-cells of C' have positive ψ'-orientation.

(2) ⇒ (3). — Let C' be a subdivision of C with a proper $(d+2)$-colouring ψ' of the 1-skeleton of C' such that all d-cells have the same ψ'-orientation (say, positive ψ'-orientation). We show that for every $(d-2)$-cell f in C', the number of incident $(d-1)$-cells is divisible by three.

Let $c = t_0 f_1 t_1 f_2 t_2 \dots t_{k-1} f_k t_0$ be the cyclic ordering of the $(d-1)$-cells $f_1, \dots, f_k$ and d-cells $t_0, t_1, \dots, t_{k-1}$ around f induced by the embedding of C' in $\mathbb{S}^d$. Let w_i be the vertex of f_i that is not in f, for $i = 1, \dots, k$. Let c_1, c_2, c_3 be the three colours that are not colours of f'. Without loss of generality, $\psi'(w_1) = c_1$ and $\psi'(w_2) = c_2$. Since all d-cells $t_0, t_1, \dots, t_{k-1}$ have the same ψ'-orientation, it must hold that $\psi'(w_3) = c_3$, $\psi'(w_4) = c_1$, $\dots$, $\psi'(w_k) = c_3$, $\psi'(w_1) = c_1$. Then it follows that k is divisible by three and hence the number of $(d-1)$-cells that are incident with f' is divisible by three.

(3) ⇒ (1). — Let C' be a subdivision of C such that for every $(d-2)$-cell, the number of incident $(d-1)$-cells is divisible by three. It suffices to show that there exists a $(d+2)$-colouring of the 1-skeleton of C'. We fix an arbitrary ordering $v_0 < v_1 < \cdots < v_n$ of the vertices $V = \{v_0, v_1, \dots, v_n\}$ of C'. Let $X = \Sigma_{d+2}$ be the symmetric group symmetric group whose elements are the permutations on $\{0, 1, 2, \dots, d+1\}$. For a d-cell t with vertices $u_0 < u_1 < \cdots < u_d$ and an element $\pi_t \in X$, the colouring (t, π_t) of t induces a colouring of the vertices of t, i.e. vertex u_i gets colour $\pi_t(i)$ (and no vertex of t gets colour $\pi_t(d+1)$).

We will use the Local-Global Colouring Lemma (Lemma 1.3) to show the implication (3) ⇒ (1). For this, we construct a proper canonical local

$(d+1)$-colouring of C, as defined in Definition 3.5. First, we need to construct the bijective maps $g_{\vec{e}}\colon X \to X$ on the directed edges $\vec{e}$ of the dual graph G. Then it suffices to show that these bijective maps satisfy Definition 3.5 (1) and (2). By Lemma 1.3, we then get a proper $(d+1)$-colouring of the 1-skeleton of C.

Let f be the $(d-1)$-cell in C' corresponding to the edge $\vec{e} = st \in E(G)$ in the dual graph G' of C'. For each colouring (s, π_s) of s (with $\pi_s \in X$) there exists a unique colouring (t, π_t) of t that agrees with (s, π_s) on the vertices of f and satisfies $\pi_s(d+1) \neq \pi_t(d+1)$; and vice versa. We define $g_{\vec{e}}$ to be the bijective map that maps each colouring $\pi_s \in X$ to the according π_t. Observe that $g_{\overleftarrow{e}} = g_{\vec{e}}^{-1}$ for all directed edges $\vec{e} \in E(G)$.

Claim 5.3. — *For each $(d-2)$-cell f in C', the f-cycle c in G' is balanced.*

Proof of Claim. — Let $c = s_0\vec{e}_1 s_1 \vec{e}_2 s_2 \ldots s_{k-1}\vec{e}_k s_0$ be the f-cycle in G and f_i be the $(d-1)$-cell in C corresponding to the edge $\vec{e}_i$ in G, for $i = 1, \ldots, k$. We have to show that $g_c = g_{\vec{e}_k} \circ \cdots \circ g_{\vec{e}_2} \circ g_{\vec{e}_1} = \mathbb{1}$.

By definition of $g_{\vec{e}_i}$, the colouring $(s_{i-1}, \pi_{s_{i-1}})$ of s_{i-1} and the colouring $(s_i, \pi_{s_i} = g_{\vec{e}_i}(\pi_{s_{i-1}}))$ of s_i agree on the vertices of f_i and therefore on the vertices of f. Let (s_0, π_{s_0}) be an arbitrary colouring of s_0. Define $\pi_{s_i} := (g_{\vec{e}_i} \circ \cdots \circ g_{\vec{e}_1})(\pi_{s_0})$, which gives a colouring (s_i, π_{s_i}) of s_i. Then all colourings (s_i, π_{s_i}) agree on the vertices of the $(d-2)$-cell f. Without loss of generality, f is coloured with the colours $\{3, 4, \ldots, d+1\}$. For $i = 1, \ldots, k$, let w_i be the vertex of f_i that is not a vertex of f. Then the w_i are coloured $0, 1, 2$ (in that cyclic order) or $0, 2, 1$ (in that cyclic order). Since every f-cycle has length divisible by three it follows that $g_c(\pi_{s_0}) = \pi_{s_0}$ for every $\pi_{s_0} \in X$. Therefore, $g_c = \mathbb{1}$ for every f-cycle c in G. □

Note that Definition 3.5 (1) is fulfilled by Claim 5.3, and Definition 3.5 (2) is fulfilled by the definition of the functions $g_{\vec{e}}$. We use Lemma 1.3 with $k = d+2$ to obtain a $(d+2)$-colouring of the 1-skeleton of C. □

5.1. Nonsubdividable triangulations

By the four-colour theorem and by Theorem A, every plane triangulation is subdividable, i.e. has a subdivision such that every vertex has degree divisible by three. For $d \geqslant 3$ there exist triangulations of $\mathbb{S}^d$ that are not subdividable, for example the triangulation of $\mathbb{S}^d$ from [23, Example 0.6] whose 1-skeleton is the complete graph K_{d+3}.

Example 5.4 ([23, Example 0.6]). — Let $d \geqslant 3$ and $k \geqslant 1$. There exists a triangulation of $\mathbb{S}^d$ whose 1-skeleton is the complete graph K_{d+k}.

Proof. — Let $C := C_{d+1}(d+k)$ be the cyclic $(d+1)$-polytope on $d+k$ vertices [23, Example 0.6]. By [23, Corollary 0.8], every vertex of C lies on the boundary of C and every pair of vertices in C forms an edge. Let C' be the boundary complex of C. Then C' is a triangulation of $\mathbb{S}^d$ whose 1-skeleton is the complete graph K_{d+k}. □

The following corollary follows from Example 5.4 and Theorem A.

Corollary 5.5. — *For every $d \geqslant 3$ there exists a triangulation of $\mathbb{S}^d$ that is not subdividable.* □

5.2. Complexity

The problem of deciding whether the 1-skeleton of a given triangulation of $\mathbb{S}^d$ is $(d+1)$-colourable is in the complexity class P. In fact, by the result of Joswig [14] (see Theorem 1.2), an algorithm only needs to check whether every $(d-2)$-cell is incident with an even number of $(d-1)$-cells.

Let us consider the problem of deciding $(d+2)$-colourability of the 1-skeleton of a given triangulation of $\mathbb{S}^d$. For the case $d = 2$ and by the work on the four-colour theorem, a quadratic time algorithm for four-colouring planar graphs has been developed [21]. For every $d \geqslant 3$ however, we show that the problem of deciding whether the 1-skeleton of a given triangulation of $\mathbb{S}^d$ is $(d+2)$-colourable is NP-complete. As an intermediate step, we prove the following lemma.

Lemma 5.6. — *It is NP-complete to decide whether the 1-skeleton of a given triangulation of $\mathbb{S}^3$ (without parallel edges) is 5-colourable.*

Proof. — By [8], it is NP-complete to decide whether a given planar 2-connected graph is 3-colourable. We reduce this problem to the problem of deciding whether the 1-skeleton of a given triangulation of $\mathbb{S}^3$ is 5-colourable.

Let $G = (V, E)$ be a planar 2-connected graph together with an embedding of G in $\mathbb{S}^2$ (which can be determined in linear time [13]). Let F be the set of faces determined by the embedding of G. Since G is 2-connected, every face in F is bounded by a cycle.

From G we first build a simplicial 2-complex C as follows. First, we add two vertices v_1 and v_2. Then we add edges uv_i for all $u \in V(G)$ and $i \in \{1, 2\}$. Moreover, we add for every edge $uu' \in E(G)$ two 2-cells $uu'v_1$

and $uu'v_2$. Let C be the resulting 2-complex. Then C can be embedded in $\mathbb{S}^3$ as follows. Let $\iota\colon G \hookrightarrow \mathbb{S}^2$ be an embedding of G into $\mathbb{S}^2$, and let $\iota'\colon \mathbb{S}^2 \hookrightarrow \mathbb{S}^3, (x,y,z) \mapsto (x,y,z,0)$ be the standard embedding of $\mathbb{S}^2$ into $\mathbb{S}^3$. Then $\mathbb{S}^3 \setminus \iota'(\mathbb{S}^2)$ determines two path-connected components, which we will refer to as the "inside" and the "outside" of the embedding of $\mathbb{S}^2$ into $\mathbb{S}^3$. First, we embed G in $\mathbb{S}^3$ using $\iota' \circ \iota$. Then we embed v_1 and every edge and every 2-cell incident with v_1 into the inside, and we embed v_2 and every edge and every 2-cell incident with v_2 into the outside of the embedding of $\mathbb{S}^2$ into $\mathbb{S}^3$.

For a face $f \in F$, let s_f be the unique chamber of C with $\iota'(f) \subseteq s_f$. Note that the map that maps each $f \in F$ to the chamber s_f of C is a bijection between the set F and the set of chambers of C. Moreover, observe that for every $f \in F$, the chamber s_f is bounded by the vertices v_1, v_2 and the vertices on the boundary of f in G.

For each face $f \in F$, whose boundary consists of the vertices $u_1u_2\dots u_k$ in that cyclic order say, we do all of the following. First, we add two vertices x_f and y_f. Moreover, we add all edges u_ix_f and u_iy_f for $i \in [k]$, and the edges v_1x_f, x_fy_f and y_fv_2. Now, we add the 2-cells on the vertices $u_iu_{i+1}x_f, u_iu_{i+1}y_f, v_1x_fu_i, x_fy_fu_i$ and $y_fv_2u_i$ for every $i \in [k]$, and the 3-cells $u_iu_{i+1}v_1x_f$, $u_iu_{i+1}x_fy_f$ and $u_iu_{i+1}y_fv_2$ for $i \in [k]$. This yields a 3-complex C'. Note that the embedding of C in $\mathbb{S}^3$ can be extended to an embedding of C' in $\mathbb{S}^3$ by placing each x_f and y_f and all edges, 2-cells and 3-cells incident with x_f or y_f into the chamber s_f of C. In fact, it is straightforward to check that C' is a triangulation of $\mathbb{S}^3$. Observe that C' does not have parallel edges. Moreover, the triangulation C' can be constructed from G in polynomial time.

It remains to show that G is 3-colourable if and only if the 1-skeleton of C' is 5-colourable. If G has a proper 3-colouring $g\colon V(G) \to [3]$, we extend g to a proper 5-colouring $c\colon V(C') \to [5]$ of the 1-skeleton of C' by letting $c(v_1) := c(y_f) := 4$ for every $f \in F$, and $c(x_f) := c(v_2) := 5$ for every $f \in F$. Conversely, let $c\colon V(C') \to [5]$ be a proper 5-colouring of the 1-skeleton of C'. If G is bipartite then we are done. Otherwise there is a face $f \in F$ on an odd number of vertices. So the vertices on the boundary of f receive three distinct colours by c. Note that x_f and y_f are adjacent to v_1 and v_2, respectively, and to all vertices on the boundary of f. So if $c(v_1) = c(v_2)$ then $c(x_f) = c(y_f)$, a contradiction. Therefore, say $4 = c(v_1) \neq c(v_2) = 5$. Since v_1 and v_2 are both adjacent to every vertex of G in C', the vertices of G are coloured with $1, 2$ and 3 only. □

Proposition 5.7. — *Let $d \geqslant 3$ be an integer. It is NP-complete to decide whether the 1-skeleton of a given triangulation of $\mathbb{S}^d$ is $(d+2)$-colourable.*

Proof. — We proceed by induction on d. For $d = 3$ the statement follows from Lemma 5.6. Assume that for some $d \geqslant 3$, it is NP-complete to decide whether a given triangulation of $\mathbb{S}^d$ is $(d+2)$-colourable. Let C be a triangulation of $\mathbb{S}^d$ and C' be the double-cone of C, i.e. C' is a triangulation of $\mathbb{S}^{d+1}$ with $V(C') = V(C) \dot\cup \{x, y\}$ and $E(C') = E(C) \cup \{xu, yu \mid u \in V(C)\}$. Then C is $(d+2)$-colourable if and only if C' is $((d+1)+2)$-colourable. □

6. Proof of the "moreover"-part of Theorem A

Given a graph G and an edge-colouring ψ, we say that a triangle in G is *monochromatic* if all of its edges have the same colour in ψ. Let $R_k(3)$ be the smallest integer $n \in \mathbb{N}$ such that every k-edge-colouring of K_n contains a monochromatic triangle. It is known that $R_2(3) = 6$ and $R_3(3) = 17$.

Lemma 6.1. — *Every $(R_k(3) - 1)$-colourable graph has a k-edge-colouring without monochromatic triangles.*

Proof. — Let $n := R_k(3)-1$. Let G be a graph with a proper n-colouring $\varphi\colon V(G) \to [n]$. Let $\psi\colon E(K_n) \to [k]$ be a k-edge-colouring of K_n without monochromatic triangles. We define a k-edge-colouring $\psi'\colon E(G) \to [k]$ by assigning to the edge $e = uv$ the colour $\psi'(e) := \psi(\{\varphi(u), \varphi(v)\})$.

We need to show that every triangle $u_1e_1u_2e_2u_3e_3u_1$ has two edges of distinct colours. Since φ is a proper n-colouring of G, we have that $\varphi(u_1)\varphi(u_2)\varphi(u_3)$ is a triangle in K_n. Since ψ is an edge-colouring of K_n without monochromatic triangles, we have without loss of generality

$$\psi(\{\varphi(u_1), \varphi(u_2)\}) \neq \psi(\{\varphi(u_2), \varphi(u_3)\}) \ .$$

Hence $\psi'(e_1) \neq \psi'(e_2)$. □

Corollary 6.2. — *Every triangulation of $\mathbb{S}^3$ whose 1-skeleton is $(R_k(3)-1)$-colourable has a k-edge-colouring without monochromatic faces.*

We prove the "moreover"-part of Theorem A by showing the following stronger result.

Theorem 6.3. — *Let C be a triangulation of $\mathbb{S}^3$. Then the following statements are equivalent.*

(1) *The 1-skeleton of C is* 5*-colourable.*

(2) *Every subdivision of C has a 2-edge-colouring without monochromatic*[4] *faces.*
(3) *The maximal subdivision C' of C (that is obtained from C by subdividing every chamber) has a 2-edge-colouring without monochromatic faces.*
(4) *The triangulation C has a 2-edge-colouring such that on the boundary of each 3-cell, every colour forms a path of length 3.*
(5) *There exists a subdivision C' of C such that for every edge in C', the number of incident faces is divisible by three.*

Proof.

(1) ⇒ (2). — Let C be a triangulation of $\mathbb{S}^3$ whose 1-skeleton is 5-colourable. Then the 1-skeleton of any subdivision C' of C is again 5-colourable. Since $R_2(3) = 6$, we can construct a 2-edge-colouring of C' without monochromatic faces by Corollary 6.2.

(2) ⇒ (3). — It is trivial.

(3) ⇒ (4). — Let C be a triangulation of $\mathbb{S}^3$ and let C' be maximal subdivision of C. Let $\psi\colon E(C') \to \{\text{red}, \text{blue}\}$ be a 2-edge-colouring of C' without monochromatic faces. Obviously, $\psi|_{E(C)}$ is a 2-edge-colouring of C without monochromatic faces. Let t be a 3-cell of C on the vertices u_1, u_2, u_3, u_4. Let $E(t)$ be the set of edges on the boundary of t, i.e. $E(t) := \{u_i u_j \mid 1 \leqslant i < j \leqslant 4\}$. It suffices to show that both colours appear exactly three times in $E(t)$.

We assume for contradiction that one color, say red, appears less than three times in $E(t)$. If red appears in at most one edge of $E(t)$, then one can find a monochromatic face on the boundary of t, a contradiction. So, red appears twice and blue appears four times in $E(t)$. Since we do not have monochromatic faces on the boundary of t, the red edges in $E(t)$ must form a matching, say $\psi(u_1u_2) = \psi(u_3u_4) = \text{red}$. Let v be the vertex of C' that was added when subdividing t. Since the edges of the triangle u_1u_2v must contain both colours and since $\psi(u_1u_2) = \text{red}$, we have without loss of generality $\psi(u_1v) = \text{blue}$. Since the triangles u_1u_4v and u_1u_3v must contain both colours and since we have $\psi(u_1u_4) = \psi(u_1v) = \psi(u_1u_3) = \text{blue}$, we must have $\psi(u_4v) = \psi(u_3v) = \text{red}$. But then, the triangle u_3u_4v is monochromatic, a contradiction.

(4) ⇒ (5). — Let $\psi\colon E(C) \to \{\text{red}, \text{blue}\}$ be a 2-edge-colouring such that on the boundary of each 3-cell, every color forms a path. We say that a 3-cell t is *even* if the order of the vertices along the red path on the boundary

[4] A face is *monochromatic* if all edges on its boundary have the same colour.

of t induces a positive orientation of t; otherwise we say that t is *odd*. Let C' be the triangulation of $\mathbb{S}^3$ that is obtained from C by subdividing every odd 3-cell. We show that for every edge in C', the number of incident faces is divisible by three.

First, observe that there exists a (unique) way to extend the 2-edge-colouring of C to a 2-edge-colouring $\psi'\colon E(C') \to \{\text{red}, \text{blue}\}$ of C' such that on the boundary of each 3-cell, every color forms a path. Indeed, let $v_0v_1v_2v_3$ be an order of the vertices of a subdivided 3-cell t along the red path and let v be the vertex that is added when subdividing t. Then we put $\psi'(vv_0) = \psi'(vv_3) = \text{red}$ and $\psi'(vv_1) = \psi'(vv_2) = \text{blue}$. Then it is straightforward to check that for each of the four 3-cells of C' included in t, every colour forms a path of length 3 on the boundary of the respective 3-cell. Moreover, the following fact is also straightforward to check, see Figure 6.1.

(6.1) All 3-cells of C' are even with respect to the 2-edge-colouring ψ'.

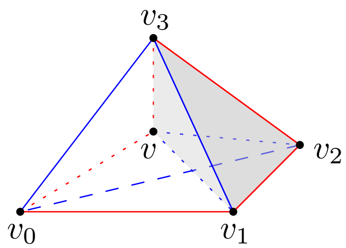

Figure 6.1. If the 3-cell t of C on the red path $v_0v_1v_2v_3$ is odd, then the 3-cell t' of C' on the red path $v_1v_2v_3v$ is even.

Now, we show that for every edge e in C', the number of incident faces is divisible by three. Let $c = t_0f_1t_1f_2t_2 \dots t_{k-1}f_kt_0$ be the cyclic ordering of the faces $f_1, \dots, f_k$ and 3-cells $t_0, t_1, \dots, t_{k-1}$ around e induced by the embedding of C' in $\mathbb{S}^3$. Let w_i be the vertex of f_i that is not in e, for $i = 1, \dots, k$. Let u and v be the endvertices of e such that for all $i = 1, \dots, k$, the ordering u, v, w_i, w_{i+1} induces a positive orientation of the corresponding 3-cell. We distinguish two cases.

Assume that the edge e is red in ψ'. If $\psi'(uw_i) = \psi'(vw_i) = \text{blue}$, then we have $\psi'(uw_{i+1}) = \text{red}$ and $\psi'(vw_{i+1}) = \text{blue}$ since an ordering of the vertices along the red path (here $w_iw_{i+1}uv$) induces a positive orientation on the corresponding 3-cell by (6.1). Analogously, if $\psi'(uw_i) = \text{red}$ and

$\psi'(vw_i)$ = blue then $\psi'(uw_{i+1})$ = blue and $\psi'(vw_{i+1})$ = red. Analogously, if $\psi'(uw_i)$ = blue and $\psi'(vw_i)$ = red then $\psi'(uw_{i+1}) = \psi'(vw_{i+1})$ = blue. Therefore, if e is red then the number of incident faces is divisible by three.

Now, assume that the edge e is blue in ψ'. This case is similar to the previous case. If $\psi'(uw_i) = \psi'(vw_i)$ = red, then we have $\psi'(uw_{i+1})$ = red and $\psi'(vw_{i+1})$ = blue since an ordering of the vertices along the red path (here vw_iuw_{i+1}) induces a positive orientation on the corresponding 3-cell by (6.1). Analogously, if $\psi'(uw_i)$ = red and $\psi'(vw_i)$ = blue then $\psi'(uw_{i+1})$ = blue and $\psi'(vw_{i+1})$ = red. Analogously, if $\psi'(uw_i)$ = blue and $\psi'(vw_i)$ = red then $\psi'(uw_{i+1}) = \psi'(vw_{i+1})$ = red. Therefore, if e is blue then the number of incident faces is divisible by three.

(5) ⇒ (1). — It follows from Theorem A. □

Corollary 6.4. — *Let C be a triangulation of $\mathbb{S}^3$ whose 1-skeleton is the complete graph K_6, and let C' be the maximal subdivision of C. Then every 2-edge-colouring of C' has a monochromatic face.*

We complement Corollary 6.4 by showing that every simplicial complex embedded in $\mathbb{S}^3$ has a 4-edge-colouring without monochromatic faces. Additionally, we ask whether any triangulation of $\mathbb{S}^3$ has a 3-edge-colouring without monochromatic faces, see Question 8.1.

Recall that the *link graph* of a simplicial 2-complex C at a vertex v is the graph L whose vertices are the edges incident to v in C, and two vertices $e_1, e_2 \in E(C)$ share an edge in L if e_1, e_2 viewed as edges in C share a face in C. If the simplicial complex C can be embedded in $\mathbb{S}^3$, then the link graph at each vertex of C is planar.

Proposition 6.5. — *Let C be a simplicial 2-complex embedded in $\mathbb{S}^3$. Then C has a 4-edge-colouring without monochromatic faces.*

Proof. — We prove this proposition by induction on the number of vertices n of C. If $n = 1$, then the statement trivially holds. Hence, let $n \geqslant 2$ and let v be a vertex of C. Let $C - v$ be the simplicial complex where we remove all cells (i.e. vertices, edges and faces) that contain v. Then, $C - v$ has a 4-edge-colouring without monochromatic faces by the induction hypothesis.

Consider the link graph L of C at v. Since L is planar, it has a 4-colouring φ by the four-colour theorem [1]. We extend the 4-edge-colouring of $C - v$ to C by giving every edge e that contains the vertex v the colour $\varphi(e)$.

We need to show that C does not contain a monochromatic face. If the face f does not contain the vertex v, it is also a face in $C - v$ and therefore not monochromatic. Hence, assume that the face f contains the vertex v, i.e. f is bounded by the edges e_1, e_2, e_3 such that e_1 and e_2 are incident to v. Then, e_1 and e_2 are adjacent vertices in the link graph L and therefore receive distinct colours. □

7. Edge and face colourings

In this section, we show how to obtain an edge colouring from a vertex colouring of a simplicial complex. An edge-colouring of a simplicial complex C is *proper* if for every face (i.e. 2-cell) f, every two distinct edges e_1 and e_2 adjacent in f receive distinct colours.

In order to derive a proper edge-colouring from a vertex-colouring, we use the following fact about 1-factorizations of complete graphs. A 1-*factorization* of a graph G is a partition of its edge set into perfect matchings. It is well-known that for every positive integer n, the complete graph K_{2n} has a 1-factorization, see [19] for example.

Proposition 7.1. — *Let k be a positive integer and C be a simplicial complex whose 1-skeleton has a proper $2k$-colouring. Then C can be edge-coloured with $2k-1$ colours.*

Proof. — Let $\varphi\colon V(C) \to [2k]$ be a $2k$-colouring of the 1-skeleton of C. Let $E(K_{2n}) = M_1 \dot\cup M_2 \dot\cup \cdots \dot\cup M_{2k-1}$ be a 1-factorization of the complete graph K_{2n} on the vertex set $[2k]$. To each edge uv of C we assign the colour i with $\{\varphi(u), \varphi(v)\} \in M_i$. Since $\varphi(u) \neq \varphi(v)$ for all edges uv in C, this is a well-defined edge-colouring. We denote this edge-colouring by the function $\psi\colon E(C) \to [2k-1]$. We need to show that every face $f = uvw$ satisfies $\psi(uv) \neq \psi(vw) \neq \psi(wu) \neq \psi(uv)$. Suppose for contradiction that two edges of f have the same colour, without loss of generality $\psi(uv) = \psi(vw)$. Then $\{\varphi(u), \varphi(v)\}$ and $\{\varphi(v), \varphi(w)\}$ are in the same perfect matching M_i of the 1-factorization of K_{2n}. Therefore $\varphi(u) = \varphi(w)$, a contradiction since there exists an edge uw in C. □

For $k = 2$, the converse of Proposition 7.1 holds for triangulations C of $\mathbb{S}^3$, as shown by Carmesin, Nevinson and Saunders [6], i.e. C has a proper 3-edge-colouring if and only if its 1-skeleton has a proper 4-colouring. However, the converse of Proposition 7.1 does not hold in general, even for triangulations of $\mathbb{S}^3$. By [23, Example 0.6], for every $n \geqslant 4$ there exists a triangulation of $\mathbb{S}^3$ whose 1-skeleton is the complete graph K_n, but which is 12-edge-colourable by [17]. In particular, 13-edge-colourability of C (in this case $k = 7$) does not imply any upper bound on the chromatic number of the 1-skeleton of C.

Corollary 7.2. — *Every triangulation of $\mathbb{S}^d$ such that every $(d-2)$-cell is incident with an even number of $(d-1)$-cells is edge-colourable with d colours if d is odd, and with $d+1$ colours if d is even.*

Proof. — Follows from Theorem 1.2 and Proposition 7.1. □

Corollary 7.3. — *Every subdividable triangulation of $\mathbb{S}^d$ is edge-colourable with $d+1$ colours if d is even, and with $d+2$ colours if d is odd.*

Proof. — Follows from Theorem A and Proposition 7.1. □

For triangulations of $\mathbb{S}^3$, we can also derive a proper face-colouring from a vertex-colouring as follows. A face-colouring of a 2-complex embedded in $\mathbb{S}^3$ is *proper* if for every 3-cell, every two distinct faces on the boundary of that 3-cell receive different colours. The *face-chromatic number* of a triangulation C of $\mathbb{S}^3$ is the least positive integer k such that there exists a proper k-face-colouring of C. The face-chromatic number of a triangulation of $\mathbb{S}^3$ is at least 4 since every 3-cell (i.e. tetrahedron) is incident with four faces. For a trivial upper bound, consider the graph $G = (F, E)$ where F is the set of faces of C and two faces are adjacent in G if they are contained in the same 3-cell. This graph has maximum degree 6 which proves that the face-chromatic number of a triangulation of $\mathbb{S}^3$ is at most 7.

Proposition 7.4. — *Let C be a triangulation of $\mathbb{S}^3$ whose 1-skeleton has a 5-colouring. Then C can be face-coloured with five colours. On the other hand, there exists a triangulation of $\mathbb{S}^3$ that has no 4-face-colouring but whose 1-skeleton has a 5-colouring.*

Proof. — Let $\varphi\colon V(C) \to \mathbb{Z}_5$ be a 5-colouring of the 1-skeleton of C. Let $E(K_5) = M_1 \dot\cup M_2 \dot\cup \cdots \dot\cup M_5$ be a partition of the edge set of the complete graph K_5 (on the vertex set $\mathbb{Z}_5$) into maximum matchings (take a 1-factorization of K_6 and remove a vertex). To each face f with vertices $v_0, v_1, v_2 \in f$, we assign the colour i with $\mathbb{Z}_5 \setminus \{\varphi(v_0), \varphi(v_1), \varphi(v_2)\} \in M_i$. We denote this face-colouring by the function $\psi\colon F(C) \to \mathbb{Z}_5$. Consider two faces f_1 and f_2 on the boundary of the same 3-cell, and with vertices $v_0, v_1, v_2 \in f_1$ and $v_1, v_2, v_3 \in f_2$. Assume that $\psi(f_1) = \psi(f_2)$, then $\{\varphi(v_0), \varphi(v_1), \varphi(v_2)\} = \{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ by the definition of ψ. It follows that $\varphi(v_0) = \varphi(v_3)$, which is a contradiction since v_0 and v_3 are adjacent in the 1-skeleton of C.

Consider the triangulation C of $\mathbb{S}^3$ whose 1-skeleton is the complete graph K_5, i.e. C is obtained from the tetrahedron by subdividing one 3-cell. Obviously, the 1-skeleton of C is 5-colourable. On the other hand observe that the triangles of K_5 are the faces of C. Hence C has $\binom{5}{3} = 10$ faces. Assume that C is 4-face-colourable. Then there exist three faces f_1, f_2, f_3 in C that have the same colour. But then, two of them share an edge and therefore lie on the boundary of the same 3-cell of C (since every edge in C is incident with exactly three faces), a contradiction. □

COROLLARY 7.5. — *Every subdividable triangulation of $\mathbb{S}^3$ can be face-coloured with five colours and this is best possible.*

Proof. — Follows from Theorem A and Proposition 7.1. □

8. Outlook

Question 1.1 remains open for $k \geqslant 3$:

QUESTION 1.1. — *For $k, d \geqslant 3$, can you find a structural characterisation of the triangulations of $\mathbb{S}^d$ whose 1-skeleton is $(d+k)$-colourable?*

By Proposition 6.5, every triangulation of $\mathbb{S}^3$ has a 4-edge-colouring without monochromatic faces. By Corollary 6.4 there exists a triangulation of $\mathbb{S}^3$ such that every 2-edge-colouring has a monochromatic face.

QUESTION 8.1. — *Is there a triangulation of $\mathbb{S}^3$ such that every 3-edge-colouring has a monochromatic face?*

Note that, by Corollary 6.2 and since $R_3(3) = 17$, every triangulation of $\mathbb{S}^3$ whose 1-skeleton has a proper 16-colouring also has a 3-edge-colouring without monochromatic faces.

Acknowledgements

I am grateful to Johannes Carmesin and Jan Kurkofka for stimulating discussions. I thank the anonymous referees for identifying errors in the initial version of this paper and drawing my attention to [23, Example 0.6].

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Manuscript received 24th November 2024,
accepted 7th November 2025.

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Innovations in Graph Theory is a member of the Mersenne Center for Open Publishing
ISSN: 3050-743X