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SUBGRAPHS WITH A POSITIVE MINIMUM SEMIDEGREE IN DIGRAPHS WITH LARGE OUTDEGREE

by Andrzej GRZESIK, Vojtěch RÖDL & Jan VOLEC (*)

Abstract. — We prove that every n-vertex directed graph G with the minimum outdegree $\delta^+(G) = d$ contains a subgraph H satisfying

$$\min\left\{\delta^+(H), \delta^-(H)\right\} \geqslant \frac{d(d+1)}{2n}.$$

We also show that if $d = o(n)$ then this bound is asymptotically best possible.

1. Introduction

A well known result of Erdős [5] states that every undirected graph with n vertices and m edges contains a subgraph with minimum degree at least $\frac{m}{n}$. In order to establish a meaningful analogue of this statement in the setting of directed graphs, one needs to impose some additional assumptions; indeed every subgraph of a transitive tournament contains a vertex with zero indegree (and similarly a vertex with zero outdegree). Huang et al. [7] proved that if a directed graph with n vertices and m arcs is Eulerian, i.e., if every vertex has its indegree equal to its outdegree, then one can always find an Eulerian subgraph with the minimum indegree at least $\frac{m^2}{24n^3}$.

In this paper, we focus on directed graphs where every vertex has a large outdegree. It is easy to observe that if every vertex in a finite digraph has non-zero outdegree, then it has to contain a directed cycle. An immediate corollary of this observation is that such a digraph contains a subgraph

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where both the minimum indegree and the minimum outdegree are non-zero. Our aim is to study a quantitative strengthening of the aforementioned corollary.

1.1. Notation

Given a finite vertex-set V, a *directed graph*, or *digraph* for short, is a pair (V, A), where $A \subseteq V \times V$. The elements of A are called *arcs*, and for a given arc $e = (u, w)$, where $u, w \in V$, we say that e is *oriented* from u to w. We note that all the digraphs in this paper are *loopless*, i.e., $A \subseteq V \times V \setminus \{(v, v) : v \in V\}$, however, pairs of different vertices are allowed to be connected by two arcs with the opposite directions. For a systematic study of digraphs, we refer the reader to the monograph of Bang-Jensen and Gutin [2].

Given a digraph $G = (V, A)$ and a vertex $v \in V$, we define the *outneighborhood of* v and *inneighborhood of* v to be

$$N_G^+(v) := \{w \in V : (v, w) \in A\} \quad \text{and} \quad N_G^-(v) := \{u \in V : (u, v) \in A\},$$

respectively. We set $\deg_G^+(v) := |N_G^+(v)|$ and $\deg_G^-(v) := |N_G^-(v)|$, and call the respective quantities the *outdegree of* v and the *indegree of* v. We drop the subscript in cases when the digraph G is clear from the context.

For a digraph G on the vertex-set V, we define the *minimum outdegree of* G, which we denote by $\delta^+(G)$, to be the smallest outedgree of a vertex in G. In other words, $\delta^+(G) := \min_{v \in V} \deg_G^+(v)$. Analogously, we define the *minimum indegree of* G as $\delta^-(G) := \min_{v \in V} \deg_G^-(v)$. Finally, the *minimum semidegree of* G, which we denote by $\delta^\pm(G)$, is the minimum of $\delta^+(G)$ and $\delta^-(G)$.

1.2. Our results

Using the notation from Section 1.1, the observation from the beginning can be reformulated as follows: every digraph G with non-zero $\delta^+(G)$ contains a subgraph H with $\delta^\pm(H) \geqslant 1$. It is natural to ask whether a stronger lower bound on $\delta^+(G)$ guarantees a stronger bound on $\delta^\pm(H)$. The following example shows that even an assumption $\delta^+(G) = \sqrt{2n} - 2$ is not sufficient to guarantee a subgraph H in G with $\delta^\pm(H) \geqslant 2$:

(1) Partition the set $[n]$ into two sets $A := \left[\sqrt{2n}\right]$ and $B := [n] \setminus A$.
(2) Orient all the pairs $i < j$ in A from j to i.

(3) For every $i \in A$, select $\sqrt{2n} - i - 1$ outneighbors of i in B such a way that every $v \in B$ has at most one inneighbor in A; note that this can be done since $\sum_{i=1}^{\sqrt{2n}} \sqrt{2n} - i - 1 = n - \sqrt{4.5n} < |B|$.
(4) Orient all the remaining pairs between A and B from B to A.

A moment of thought reveals that every vertex has at least $\sqrt{2n} - 2$ outneighbors, yet every subgraph H satisfies $\delta^{\pm}(H) \leqslant 1$; see Section 3 for details. The main result of this paper is that a lower bound $\delta^{+}(G) \geqslant \sqrt{2n}$ already yields a subgraph H with $\delta^{\pm}(H) > 1$.

Theorem 1.1. — *Every loopless digraph G with n vertices and outdegree $\delta^{+}(G) \geqslant d$ has a subgraph H with $\delta^{\pm}(H) \geqslant \frac{d(d+1)}{2n}$.*

The rest of the paper is organized as follows. We prove Theorem 1.1 in Section 2. In Section 3, we generalize the construction from the previous paragraph and show that Theorem 1.1 is for $d = o(n)$, up to lower order terms, best possible. We conclude by Section 4, where we discuss the situation for the dense case $d = \alpha n$, and state some related open problems.

2. Proof of Theorem 1.1

Fix an n-vertex digraph G. Without loss of generality, we will assume that every vertex of G has the outdegree equal to d, as otherwise we pass to an appropriate spanning subgraph of G. Now consider the following greedy procedure of one-by-one removing vertices from G:

(1) Let $G_n := G$.
(2) For $i = n, n-1, \ldots, 2, 1$, do:
  (a) Let v_i be a vertex in G_i with $\min\{\deg^{+}_{G_i}(v_i), \deg^{-}_{G_i}(v_i)\} = \delta^{\pm}(G_i)$.
  (b) Set $G_{i-1} := G_i - v_i$.

We note that in case of a tie between the minimum indegree and the minimum outdegree during the i-th step, i.e., when $\delta^{+}(G_i) = \delta^{-}(G_i)$, we choose as v_i a vertex that has the smallest indegree.

For the rest of the proof, we will be considering the vertices of G being ordered as $v_1 < v_2 < \ldots < v_n$. In other words, it is the reversed order with respect to the removal during the greedy procedure. For brevity, given a vertex $v \in V(G)$, we will be writing $N^{+}_{L}(v)$ and $N^{+}_{R}(v)$ to denote the set of the outneighbors of v in G that are in the order before v and after v, respectively. Analogously, we define N^{-}_{L} and N^{-}_{R}. In other words,

$$N^{+}_{L}(v) := \left\{w \in N^{+}_{G}(v) : w < v\right\}, \quad N^{+}_{R}(v) := \left\{w \in N^{+}_{G}(v) : w > v\right\},$$
$$N^{-}_{L}(v) := \left\{w \in N^{-}_{G}(v) : w < v\right\}, \quad N^{-}_{R}(v) := \left\{w \in N^{-}_{G}(v) : w > v\right\}.$$

Let $V^- := \{v_i : \deg^-_{G_i}(v_i) = \delta^{\pm}(G_i)\}$, and $V^+ := V(G) \setminus V^-$. In words, the set V^- consists of all the vertices that the greedy procedure removed because of their small indegree, and the vertices in V^+ are those that were removed because of their small outdegree. Set $c := \max_{i\in[n]} \delta^{\pm}(G_i)$. Clearly, it suffices to show that $c \geqslant \frac{d(d+1)}{2n}$. We will do so by distinguishing two cases based on the size of V^+.

We first focus on the case $|V^+| \geqslant d$. Let X be the last d vertices from V^+, i.e., the first d vertices that the greedy procedure removed because of having the smallest outdegree in the corresponding subgraph. Since every vertex of the digraph G has outdegree equal to d, it holds that $d^2 = \sum_{x\in X} \deg^+_G(x)$. On the other hand, every vertex $x \in X$ satisfies

$$\deg^+_G(x) = |N^+_L(x)| + |N^+_R(x) \cap X| + |N^+_R(x) \setminus X| \,.$$

By the definition of $X \subseteq V^+$, every $x \in X$ also satisfies $|N^+_L(x)| \leqslant c$. Furthermore, we claim that

$$\sum_{x\in X} |N^+_R(x) \cap X| \leqslant \binom{d}{2} \,. \tag{2.1}$$

Indeed, every pair of the vertices from X can contribute to this sum by at most one. Finally, the fact that X consists of the rightmost vertices of V^+ implies that $\bigcup_{x\in X} \left(N^+_R(x) \setminus X\right) \subseteq V^-$, which together with a standard double-counting argument yields that

$$\sum_{x\in X} |N^+_R(x) \setminus X| = \sum_{x\in X} |N^+_R(x) \cap V^-| = \sum_{w\in V^-} |N^-_L(w) \cap X| \leqslant (n-d)c \,.$$

Therefore, we conclude that

$$d^2 = \sum_{x\in X} |N^+_L(x)| + |N^+_R(x) \cap X| + |N^+_R(x) \setminus X| \leqslant d\cdot c + \binom{d}{2} + (n-d)c \,.$$

Rearranging the last inequality readily yields that $c \geqslant \frac{d(d+1)}{2n}$.

It remains to consider the case when $|V^+| < d$. This time, let Y^- be the first $(d - |V^+|)$ vertices from V^-, and let $Y := Y^- \cup V^+$. Firstly, observe that all the vertices to the left of a vertex $y \in Y^-$ must lie inside the set Y. In particular, $N^-_L(y) \subseteq Y$ and $N^+_L(y) \subseteq Y$ for any $y \in Y^-$. Analogously as in (2.1), it holds that

$$\sum_{y\in Y^-} |N^+_L(y)| + \sum_{y\in Y} |N^+_R(y) \cap V^+| \leqslant \binom{d}{2} \,,$$

since the left-hand side of the inequality counts every pair of the vertices from Y at most once. We also have that $|N^+_L(y)| \leqslant c$ for every $y \in V^+$, and additionally, it holds that $|N_L^-(y)| \leqslant c$ for every $y \in Y^-$. Thus,

$$\sum_{y\in V^+} |N_L^+(y)| + \sum_{y\in Y} |N_R^+(y) \cap Y^-| \leqslant |V^+| \cdot c + \sum_{w\in Y^-} |N_L^-(w)| \leqslant d \cdot c\,.$$

Finally, a double-counting argument yields that

$$\sum_{y\in Y} |N_R^+(y) \setminus Y| = \sum_{w\in V\setminus Y} |N_L^-(w) \cap Y| \leqslant (n-d)c\,.$$

Since

$$|N_R^+(y)| = \left|N_R^+(y) \cap V^+\right| + \left|N_R^+(y) \cap Y^-\right| + \left|N_R^+(y) \setminus Y\right| \text{ for all } y \in Y,$$

we have that

$$d^2 = \sum_{y\in Y} |N_R^+(y)| + \sum_{y\in Y^-} |N_L^+(y)| + \sum_{y\in V^+} |N_L^+(y)| \leqslant \binom{d}{2} + d\cdot c + (n-d)c.$$

As in the previous case, rearranging $d^2 \leqslant n \cdot c + \binom{d}{2}$ yields that $c \geqslant \frac{d(d+1)}{2n}$.

3. Tournaments with all subgraphs having small $\delta^\pm$

In this section, we prove that Theorem 1.1 is essentially best possible, even if we would consider only *tournaments*, i.e., orientations of the edge-set of complete undirected graphs.

Proposition 3.1. — *Fix two positive integers k and ℓ, and let $d := 2k\ell - 1$ and $n_0 := (k+1)d + \ell$. For every integer $n \geqslant n_0$ there exists a tournament T with n vertices and $\delta^+(T) = d$ such that*

$$\max_{H\subseteq T} \delta^\pm(H) \leqslant \ell \leqslant \left(1 + \frac{k+1}{k^2}\right) \cdot \frac{d(d+1)}{2n}.$$

Proof. — Fix the parameters k and ℓ. Firstly, let us consider the case $n = n_0$, and let V be any vertex-set of cardinality n. We partition V into three parts A, B and C of respective sizes ℓ, $k \cdot d$ and d. Let us now describe the orientations of pairs of vertices from V. Firstly,

(1) each of the sets A, B, C induces a transitive orientation,
(2) all the pairs between A and B are oriented from B to A, and
(3) all the pairs between A and C are oriented from A to C.

Clearly, $\deg_T^+(v) \geqslant |C| = d$ for every vertex $v \in A$. Now, in order to have $\deg_T^+(v) = d$ for every $v \in C$, we need to orient a total of $\binom{d+1}{2}$ pairs from C to B, where the i-th vertex of C with respect to the transitive order needs to have exactly i outneighbors in B. Our aim is to do this in such a way

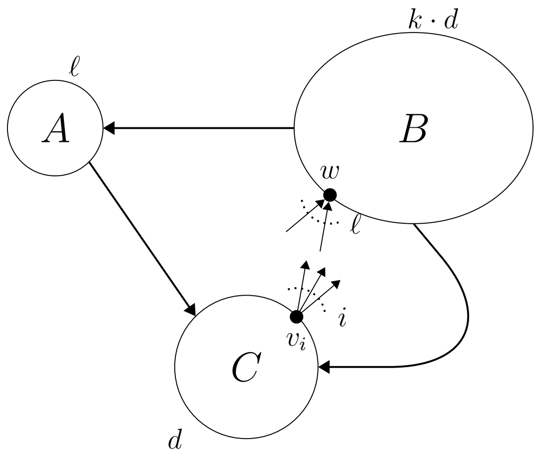

Figure 3.1. The tournament constructed in Proposition 3.1.

that every vertex of B will have exactly $\frac{d+1}{2k} = \ell$ incoming arcs from C. We claim that this is always possible. Indeed, order B arbitrarily and let S be a sequence of length $\ell \cdot |B|$ formed by ℓ copies of the ordered B placed after each other. Since $\ell|B| = \binom{d+1}{2}$ and $|B| \geqslant d$, we can go over the vertices in C one by one and orient arcs from the i-th vertex to the next i members of S. This establishes the desired $\deg_T^+(v) = d$ for every $v \in C$ as well as $|N^-(w) \cap C| = \ell$ for every $w \in B$. We orient all the remaining pairs between B and C from B to C. In particular, every vertex in B has now exactly $d - \ell + |A| = d$ outneighbors in $C \cup A$.

In the previous paragraph we have shown that $\delta^+(T) = d$, thus it remains to find an upper bound on $\delta^\pm(H)$ for any given $H \subseteq T$. Without loss of generality, we will assume that H is induced. If $V(H)$ is disjoint from B then $\delta^+(H) = \delta^-(H) = 0$, because $A \cup C$ induces a transitive tournament. On the other hand, in case $V(H) \cap B \neq \emptyset$, let v be the vertex with zero indegree in the transitive tournament induced by $V(H) \cap B$. Since $\deg_H^-(v) \leqslant \left|N_T^-(v) \cap C\right| = \ell$, we conclude that

$$\delta^\pm(H) \leqslant \ell = \frac{d+1}{2k} = \left(1 + \frac{d+\ell}{dk}\right) \cdot \frac{d(d+1)}{2n} \leqslant \left(1 + \frac{k+1}{k^2}\right) \cdot \frac{d(d+1)}{2n}.$$

It remains to consider the case $n > n_0$. Given the parameters k and ℓ, let T_0 be the previously constructed tournament on n_0 vertices. In order to construct the desired tournament T, add to T_0, one by one, a total of $n - n_0$ vertices in such a way that every newly added vertex has the indegree zero. The obtained tournament T satisfies $\delta^+(T) = \delta^+(T_0) = d$, and any $H \subseteq T$ with $\delta^-(H) > 0$ satisfies that $H \subseteq T_0$. In particular, it holds that $\delta^\pm(H) \leqslant \ell$ for every $H \subseteq T$. □

4. Dense digraphs

If $d = o(n)$, then Theorem 1.1 and Proposition 3.1 determine, up to a multiplicative factor $(1+o(1))$, the best possible bound on $\max_{H\subseteq G} \delta^\pm(H)$ that holds for every n-vertex digraph G with $\delta^+(G) \geqslant d$. In the regime when $d = \alpha n$ for some fixed $\alpha > 0$, Theorem 1.1 guarantees the existence of a subgraph H with $\frac{\delta^\pm(H)}{n} \geqslant \frac{\alpha^2}{2}$.

On the other hand, for any rational $\alpha \leqslant \frac{1}{4}$, let us fix $k := \lfloor \frac{1}{\alpha} \rfloor - 2$ and a large enough integer D such that both $\ell := \frac{D}{2k}$ and $n := \frac{D}{\alpha}$ are integers. Set $d := 2k\ell - 1 = D - 1$ and note that for such a choice of the parameters, it holds that $(k+1)d + \ell < n$. The parameters k, ℓ and n satisfy the assumptions of Proposition 3.1, hence there exists an n-vertex tournament T with $\delta^+(T) = d = (\alpha - \frac{1}{n})n$ and $\max\limits_{H\subseteq G} \delta^\pm(H) \leqslant \ell$. Therefore, for $\alpha \leqslant \frac{1}{4}$, it holds that

$$\max_{H\subseteq G} \frac{\delta^\pm(H)}{n} \leqslant \frac{\ell}{n} = \frac{\alpha}{2\left(\lfloor \frac{1}{\alpha} \rfloor - 2\right)} \leqslant \frac{\alpha}{2\left(\frac{1}{\alpha} - 3\right)} \leqslant \frac{\alpha^2(1+12\alpha)}{2}, \tag{4.1}$$

which is very close to the bound proven in Theorem 1.1 for α small.

4.1. Very dense digraphs

We have observed that if $\alpha > 0$ is small, then the constant $\frac{\alpha^2}{2}$ in Theorem 1.1 is almost optimal; see (4.1). In this section, we prove that if $\alpha \geqslant 0.7832$ then every n-vertex digraph G with $\delta^+(G) = \alpha n$ has $H \subseteq G$ with $\frac{\delta^\pm(H)}{n} > \frac{\alpha^2}{2}$.

Let us start with introducing some additional notation. Given an integer n and $\alpha \in (0,1)$, let $\mathcal{G}_{n,\alpha}$ be the set of all n-vertex digraphs G with $\delta^+(G) \geqslant \alpha n$. Next, let

$$h(n,\alpha) := \min_{G\in\mathcal{G}_{n,\alpha}} \max_{H\subseteq G} \frac{\delta^\pm(H)}{n} \qquad \text{and} \qquad h(\alpha) := \liminf_{n\to\infty} h(n,\alpha).$$

Note that $h(\alpha) \leqslant \alpha$ by considering random digraphs of density α. Theorem 1.1 yields that $h(\alpha) \geqslant \frac{\alpha^2}{2}$ by repeatedly removing vertices of a given graph that have small indegree or outdegree. However, if $\alpha \geqslant 0.7832$ then one gets a stronger lower bound by removing only those vertices that have a small indegree. More precisely, we assert that

$$h(\alpha) \geqslant 1 - \sqrt{3 - 4\alpha + \alpha^2}\,,$$

which is larger than $\frac{\alpha^2}{2}$ for $\alpha > \sqrt{2\sqrt{2}+2} - \sqrt{2} \approx 0.7832$.

To this end, fix an n-vertex digraph $G = (V, A)$. Without loss of generality, every vertex $v \in V$ satisfies $\deg^+_G(v) = \lceil \alpha n \rceil$. For the rest of this section, we will ignore floors and ceilings since the resulting $O(1/n)$ difference has no impact on the value of $h(\alpha)$. Now remove one by one all its vertices that currently have the indegree smaller than βn for an appropriately chosen $\beta < \alpha$. This procedure terminates either when all the vertices of G have been removed, or, when the minimum indegree of the current subgraph is at least βn.

Set $\beta := 1 - \sqrt{3 - 4\alpha + \alpha^2}$. Suppose the indegree-greedy procedure described in the previous paragraph is applied to G, and let τ denote the proportion of the vertices of G that has been removed so far. Let $v_1, v_2, \dots, v_{\tau n}$ be those vertices in the order of their removal, and, for every $i \in [\tau n]$ we set $W_i := \{v_1, v_2, \dots, v_i\}$. Double counting the number of arcs of G yields that

$$\begin{aligned} \alpha &= \sum_{i=1}^{\tau n} \frac{|N^-_G(v_i) \cap W_i| + |N^-_G(v_i) \setminus W_i|}{n^2} + \sum_{v \in V \setminus W_{\tau n}} \frac{\deg^-_G(v)}{n^2} \\ &< \frac{\tau^2}{2} + \beta\tau + (1-\tau). \end{aligned} \tag{4.2}$$

Let $f(\tau) := \frac{\tau^2}{2} + \beta\tau + (1-\tau) - \alpha$. A bit tedious calculation reveals that $f(\alpha - \beta) = 0$, which together with (4.2) implies that the procedure has to terminate after removing less than $(\alpha - \beta)n$ vertices from G.

Let $H \subseteq G$ be the remaining graph after the procedure terminates, and let τ_0 be the total proportion of the removed vertices from G. In other words, H has $(1 - \tau_0)n$ vertices. It follows that every vertex of H has the indegree at least βn, and the outdegree at least $(\alpha - \tau_0)n$. Therefore,

$$\frac{\delta^{\pm}(H)}{n} \geqslant \min\{\alpha - \tau_0, \beta\} \geqslant \min\{\alpha - (\alpha - \beta), \beta\} = \beta = 1 - \sqrt{3 - 4\alpha + \alpha^2}\,.$$

In particular, it holds that $h(\alpha) \geqslant \max\left\{\frac{\alpha^2}{2}, 1 - \sqrt{3 - 4\alpha + \alpha^2}\right\}$. We believe it is an interesting open problem to find stronger estimates on $h(\alpha)$.

Problem 1. — *For a given $\alpha \in (0, 1)$, determine the value of $h(\alpha)$.*

4.2. Orientations of very dense undirected graphs

We say that a digraph G is an *oriented graph*, i.e., an orientation of the edge-set of a simple undirected graph, if every pair of the vertices of G is joined by at most one arc. For any n-vertex oriented graph G with $\delta^+(G) = \alpha n$ it must hold that $\alpha < 1/2$, so at the first glance the previous subsection does not provide anything useful for G. However, the double-counting argument (4.2) can be easily improved for oriented graphs since $\deg_G^-(v) < (1-\alpha)n$ for every vertex $v \in V(G)$. In the rest of this section, we show that this improvement yields the existence of a subgraph $H \subseteq G$ such that

$$\frac{\delta^\pm(H)}{n} \geqslant 1 - \alpha - \sqrt{3 - 8\alpha + 4\alpha^2}\,. \tag{4.3}$$

Observe that the right-hand side of (4.3) is larger than $\frac{\alpha^2}{2}$ for $\alpha > \alpha_1$, where α_1 is the only positive real root of $x^4 + 4x^3 - 16x^2 + 24x - 8$. We note that $\alpha_1 < 0.4528$.

Let us now introduce some additional notation, which will be analogous to the one in Section 4.1. For an integer n and $\alpha \in (0, \frac{1}{2})$, let $\widehat{\mathcal{G}}_{n,\alpha}$ be the set of all n-vertex oriented graphs G such that $\delta^+(G) \geqslant \alpha n$, and

$$\widehat{h}(n,\alpha) := \min_{G \in \widehat{\mathcal{G}}_{n,\alpha}} \max_{H \subseteq G} \frac{\delta^\pm(H)}{n} \qquad \text{and} \qquad \widehat{h}(\alpha) := \liminf_{n\to\infty} \widehat{h}(n,\alpha)\,.$$

Since $\widehat{\mathcal{G}}_{n,\alpha} \subseteq \mathcal{G}_{n,\alpha}$, it holds that $\widehat{h}(\alpha) \geqslant h(\alpha)$. Moreover, $\widehat{h}(\alpha) \leqslant \alpha$ by considering, for example, random subsets of the arc-set of a balanced tournament.

We are now ready to explain the existence of $H \subseteq G$ satisfying (4.3). Fix an n-vertex oriented graph $G \in \widehat{\mathcal{G}}_{n,\alpha}$. Without loss of generality, we will again assume that $\deg_G^+(v) = \lceil \alpha n \rceil$ for every $v \in V(G)$, and in the rest, we will ignore floors and ceilings. Now consider removing one by one all the vertices that have the current indegree smaller than $\widehat{\beta} n$ for some fixed $\widehat{\beta} > 0$. As we have already mentioned, every vertex $v \in V(G)$ satisfies $\deg_G^-(v) < (1-\alpha)n$. Thus the double-counting argument analogous to (4.2) yields the following:

$$\alpha < \frac{\tau^2}{2} + \widehat{\beta}\tau + (1-\tau)(1-\alpha)\,. \tag{4.4}$$

Let H be the remaining subgraph after the indegree-greedy procedure terminates, and set

$$\widehat{\beta} := 1 - \alpha - \sqrt{3 - 8\alpha + 4\alpha^2} \quad \text{and} \quad \widehat{f}(\tau) := \frac{\tau^2}{2} + \widehat{\beta}\tau + (1-\tau)(1-\alpha) - \alpha\,.$$

We claim that H satisfies (4.3). Indeed, since $\widehat{f}(\alpha-\widehat{\beta})=0$, the inequality (4.4) implies that H has $(1-\tau_0)n$ vertices for some $\tau_0<\alpha-\widehat{\beta}$. In particular,

$$\frac{\delta^{\pm}(H)}{n}\geqslant\min\{\alpha-\tau_0,\widehat{\beta}\}\geqslant\min\{\alpha-(\alpha-\widehat{\beta}),\widehat{\beta}\}=1-\alpha-\sqrt{3-8\alpha+4\alpha^2}\,.$$

We have shown that $\widehat{h}(\alpha)\geqslant\max\left\{\frac{\alpha^2}{2},1-\alpha-\sqrt{3-8\alpha+4\alpha^2}\right\}$. As in the previous subsection, we find interesting to establish stronger estimates on $\widehat{h}(\alpha)$.

Problem 2. — *For a given $\alpha\in\left(0,\frac{1}{2}\right)$, determine the value of $\widehat{h}(\alpha)$.*

4.3. Normalized minimum semidegree

Throughout this subsection, fix a large $G\in\mathcal{G}_{n,\alpha}$. A variant of the question we consider in this paper is to maximize, over all subgraphs $H\subseteq G$, the value of $\delta^{\pm}(H)$ normalized by the number of vertices of H. In other words, our aim is now to bound the expression

$$\mu(G):=\max_{\emptyset\neq H\subseteq G}\frac{\delta^{\pm}(H)}{|V(H)|}\,.$$

Let $\ell:=\lceil 1/\alpha\rceil$ and $b\in\mathbb{N}$. Recall that the b-blowup of a directed cycle of length ℓ, which we will denote by $C_\ell[b]$, is the oriented graph with $b\ell$ vertices partitioned into ℓ independent sets $V_0,V_1,\dots,V_{\ell-1}$ of size b each such that uw is an arc for $u\in V_i$ and $w\in V_j$ if and only if $j-i\equiv 1$ modulo ℓ. A moment of thought reveals that $\mu(C_\ell[b])\leqslant 1/\ell\leqslant\alpha$.

On the other hand, we claim that $\mu(G)\geqslant\frac{\alpha}{2}$ for every $G\in\mathcal{G}_{n,\alpha}$. The celebrated Caccetta–Häggkvist conjecture [3] states that G contains a directed cycle of length at most $\lceil 1/\alpha\rceil$. In particular, a consequence of the conjecture would be $\mu(G)\geqslant\frac{1}{\lceil 1/\alpha\rceil}$. A result of Chvátal and Szemerédi [4] weakens the conclusion of the conjecture and establishes a directed cycle of length less than $\frac{2}{\alpha}$, thus a subgraph $H\subseteq G$ with $\frac{\delta^{\pm}(H)}{|V(H)|}\geqslant\frac{\alpha}{2}$ always exists. Note that the multiplicative constant 2 from [4] was later improved by Shen [8].

The subgraph $H\subseteq G$ found in the previous paragraph is only of a constant order. It is natural to ask whether one can also find $H'\subseteq G$ with $\frac{\delta^{\pm}(H')}{|V(H')|}=\Theta(\alpha)$ while $|V(H')|$ is "large". The answer is yes via a relatively standard application of the regularity method for directed graphs: Given $G\in\mathcal{G}_{n,\alpha}$, partition its vertex-set using a degree-form of the regularity lemma (see, e.g., [1]), and then apply the above-mentioned result of

Chvátal and Szemerédi to the so-called cluster graph between the dense regular pairs. It follows that G contains $\Theta\left(n^{\ell}\right)$ cycles of length ℓ for some $2 \leqslant \ell \leqslant \frac{2}{\alpha-\varepsilon}$, hence a classical supersaturation result [6] yields that G contains $C_{\ell}[b]$ with $b = \mathrm{polylog}(n)$.

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Andrzej GRZESIK
Faculty of Mathematics and Computer Science
Jagiellonian University
Prof. St. Łojasiewicza 6
30-348 Kraków (Poland)
andrzej.grzesik@uj.edu.pl

Vojtěch RÖDL
Department of Mathematics
Emory University
Atlanta (USA)
vrodl@emory.edu

Jan VOLEC
Department of Theoretical Computer Science
Faculty of Information Technology
Czech Technical University in Prague
Thákurova 9
Prague, 160 00 (Czech Republic)
jan@ucw.cz

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