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INDUCED MINORS AND REGION INTERSECTION GRAPHS

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Abstract. — We show that for any positive integers g and t, there is a $K_6^{(1)}$-induced-minor-free graph of girth at least g that is not a region intersection graph over the class of K_t-minor-free graphs. This answers in a strong form the recently raised question of whether for every graph H there is a graph H' such that H-induced-minor-free graphs are region intersection graphs over H'-minor-free graphs.

1. Introduction

Inspired by the success of Robertson and Seymour's graph minor theory [18], a recent line of work aims to extend this theory to the realm of induced-minor-free classes.(1) Currently, far less is understood on classes excluding an induced minor than on those excluding a minor. While H-minor-free n-vertex graphs are known since the 90's to have treewidth $O_H(\sqrt{n})$ [2], foreshadowed a decade earlier by the Lipton–Tarjan planar separator theorem [14], only recently were H-induced-minor-free m-edge graphs shown to have treewidth $\widetilde{O}_H(\sqrt{m})$ [12].

There are several open questions (for simplicity, we phrase all of them as conjectures) on induced-minor-free classes.

- For any planar graph H, the independence number of any H-induced-minor-free graph can be computed in polynomial time (see [6, Question 8.2]).(2)

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(1) All the relevant notions are defined in Section 2.

(2) Merely obtaining a quasipolynomial-time algorithm is also a wide open question.

- For any planar graph H, every H-induced-minor-free graph admits a balanced separator dominated by a subset of size $O_H(1)$ (Gartland–Lokshtanov's conjecture [10]).
- For any planar graph H, every H-induced-minor-free graph has treewidth at most linear in its maximum degree (see [4]).
- For any graph H, the independence number admits a polynomial-time approximation scheme in H-induced-minor-free graphs.(3)
- For any planar graph H, weakly sparse H-induced-minor-free classes have bounded twin-width (a special case is mentioned in [3]).
- For any planar graph H, every H-induced-minor-free graph has treewidth at most linear in its Hadwiger number (see [5]).
- For any graph H, every H-induced-minor-free graph is quasi-isometric to an H-minor-free graph ([8, 11]).

All these questions are open within classes of large girth, a condition which may make them more approachable. One more question, posed independently by Lokshtanov [15] and McCarty [16], is whether region intersection graphs could provide a bridge between the structure of minors and induced minors. A graph G is a *region intersection graph* (*RIG*) over a graph H if there exists a collection $\mathcal{R} = (R_v \subseteq H : v \in V(G))$ of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. We call H the *host graph* of G.

Question 1.1. — *Is every graph class excluding a fixed induced minor included in the region intersection graphs of a class excluding a fixed minor?*

If true, one could then work with the host graph and benefit from its decomposition given by the Graph Minor Structure Theorem [19]. Wiederrecht asked a related question of whether one can *determine* if a given induced-minor-free class is a region intersection graph over a minor-free class [21].

Region intersection graphs were introduced by Lee [13] as a generalization of the well-studied class of string graphs (intersection graph of curves on the plane). Indeed, a graph is a string graph if and only if it is a region intersection graph over some planar graph. The class of string graphs does not exclude any graph as a minor, but excludes any 1-subdivision of a non-planar graph as an induced minor [20]. More generally, Lee [13] proved the following relationship between region intersection graphs and minors.

(3) This question has been informally discussed within the wider graph theory community.

Lemma 1.2 ([13]). — *For every graph G, if a graph H is not a minor of G then any graph that contains $H^{(1)}$ as an induced minor is not a region intersection graph over G.*

Thus RIGs over an H-minor-free class are examples of classes excluding an induced minor. The theory on region intersection graphs, and mainly on string graphs, is more advanced than that of induced-minor-free graphs. For instance, RIGs over K_t-minor-free classes can be $O_t(1)$-vertex-colored (or $O_t(1)$-edge-colored) such that every monochromatic connected component has bounded weak diameter [1, 13]. Such a result is useful in various contexts, and it would resolve several conjectures for classes excluding a fixed induced minor (see for instance [4, 12]). One way to achieve that would be via a positive answer to Conjecture 1.1.

Unfortunately, we answer Conjecture 1.1 negatively, and perhaps more surprisingly, even within classes of arbitrarily large girth.

Theorem 1.3. — *For any positive integers t and g, there is a $K_6^{(1)}$-induced-minor-free graph of girth at least g that is not in RIG($\{H : H$ is K_t-minor-free$\}$).*

The bridge between induced-minor-freeness and minor-freeness (if it exists) is not given straightforwardly by region intersection graphs. Our construction for proving Theorem 1.3 is an extension of the so-called *Pohoata–Davies grids* [7, 17] (see Figure 1.1), a key family of graphs in the study of induced subgraphs and tree-decompositions.

Hopefully, our construction steers the search for a link between induced-minor-freeness and minor-freeness in a more fortunate direction.

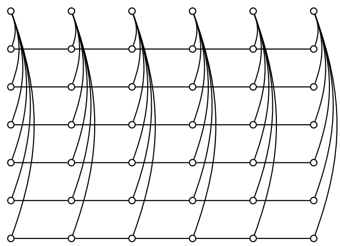

Figure 1.1. The Pohoata–Davies 6×6 *grid.*

2. Preliminaries

Given an integer i, we denote by $[i]$ the set of integers that are at least 1 and at most i.

2.1. Standard graph-theoretic notation

We denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G, respectively. A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if H can be obtained from G by vertex and edge deletions. Graph H is an *induced subgraph* of G if H is obtained from G by vertex deletions only. For $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted $G[S]$, is obtained by removing from G all the vertices that are not in S. Then $G - S$ is a short-hand for $G[V(G) \setminus S]$.

A set $X \subseteq V(G)$ is connected (in G) if $G[X]$ has a single connected component. The *girth* of a graph is the number of vertices of one of its shortest cycles, and ∞ if the graph is acyclic. A graph class is *weakly sparse* if it excludes $K_{t,t}$ as a subgraph for some finite integer t. A *balanced separator* of an n-vertex graph G is a set $X \subseteq V(G)$ such that $G - X$ has no connected component on more than $n/2$ vertices.

If G is a graph and ℓ is a positive integer, then $G^{(\ell)}$ denotes the *ℓ-subdivision* of G (replacing every edge of G by a path with $\ell + 1$ edges), and ℓG denotes the graph obtained from ℓ disjoint copies of G. We call the original vertices of $V(G)$ in $G^{(\ell)}$ *branching vertices*, and the added vertices (which have degree 2) *subdivision vertices*. We say that two disjoint sets $X, Y \subseteq V(G)$ are *anti-complete* if there is no edge in G with one end in X and the other in Y. The *diameter of G* is defined as $\max_{u,v \in V(G)} d_G(u, v)$, where $d_G(u, v)$ is the number of edges in a shortest path between u and v. The *weak diameter of S in G* for $S \subseteq V(G)$ is equal to $\max_{u,v \in S} d_G(u, v)$.

2.2. Tree-decomposition

A *tree-decomposition* of a graph G is a collection $\mathcal{T} = (W_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T, such that

- for every edge $uv \in E(G)$, some bag W_x contains both u and v, and
- for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a non-empty (connected) subtree of T.

The *width* of $\mathcal{T}$ is $\max\{|W_x|\colon x \in V(T)\} - 1$. The *treewidth* of G is the minimum width of a tree-decomposition of G. The *adhesion* of $\mathcal{T}$ is $\max\{|W_x \cap W_y|\colon xy \in E(T)\}$. The *torso* of a bag W_x (with respect to $\mathcal{T}$), denoted by $G\langle W_x\rangle$, is the graph obtained from the induced subgraph $G[W_x]$ by adding edges so that $W_x \cap W_y$ is a clique for each edge $xy \in E(T)$. A *path-decomposition* is a tree-decomposition in which the underlying tree is a path, simply denoted by the corresponding sequence of bags $(W_1, \ldots, W_n)$.

2.3. Minors, induced minors, and region intersection graphs

A graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. Equivalently, H is a minor of G if there exists a model $\mathcal{M} = (X_v \subseteq G\colon v \in V(H))$ of H in G which is a collection of disjoint connected subgraphs of G such that X_u and X_v are adjacent whenever $uv \in E(H)$. Each X_u is called a *branch set*. A graph H is an *induced minor* of a graph G if H is isomorphic to a graph that can be obtained from an induced subgraph of G by contracting edges. Equivalently, H is an induced minor of G if there is a model $\mathcal{M} = (X_v \subseteq G\colon v \in V(H))$ of H in G with the additional constraint that X_u and X_v are adjacent if and only if $uv \in E(H)$. A graph G is *H-minor-free* (resp. *H-induced-minor-free*) if H is not a minor (resp. an induced minor) of G.

Recall that a graph G is a *region intersection graph* over a graph H if there exists a collection $\mathcal{R} = (R_v \subseteq H\colon v \in V(G))$ of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. We denote by $\mathrm{RIG}(H)$ the class of graphs that are region intersection graphs over H. By extension, given a graph class $\mathcal{C}$, $\mathrm{RIG}(\mathcal{C})$ denotes the class of graphs that are region intersection graphs over some graph of $\mathcal{C}$.

2.4. Graph minor structure theorem

The Graph Minor Structure Theorem of Robertson and Seymour [19] states that every K_t-minor-free graph has a tree-decomposition with adhesion of bounded size such that each torso can be constructed using three ingredients: graphs on surfaces, vortices, and apices. To describe this formally, we need the following definitions.

Let G_0 be a graph embedded in a surface Σ. A closed disk D in Σ is G_0-*clean* if its only points of intersection with G_0 are vertices of G_0 that lie on the boundary of D. Let $x_1, \dots, x_b$ be the vertices of G_0 on the boundary of D in the order around D. A *D-vortex* (with respect to G_0) of a graph H is a path-decomposition $(W_1, \dots, W_b)$ of H such that $x_i \in W_i$ for each $i \in [b]$, and $V(G_0 \cap H) = \{x_1, \dots, x_b\}$.

For integers $g, p, a \geqslant 0$ and $k \geqslant 1$, a graph G is *(g,p,k,a)-almost-embeddable* if for some set $Z \subseteq V(G)$ with $|Z| \leqslant a$, there are graphs $G_0, G_1, \dots, G_p$ such that:

- $G - Z = G_0 \cup G_1 \cup \dots \cup G_p$,
- $G_1, \dots, G_p$ are pairwise vertex-disjoint,
- G_0 is embedded in a surface Σ of Euler genus at most g,
- there are p pairwise disjoint G_0-clean closed disks $D_1, \dots, D_p$ in Σ, and
- for $i \in [p]$, there is a D_i-vortex $(W_1, \dots, W_{b_i})$ of G_i of width at most k.

The vertices in Z are called *apex* vertices—they can be adjacent to any vertex in G. A graph is *ℓ-almost-embeddable* if it is (g,p,k,a)-almost-embeddable for some $\ell \geqslant g, p, k, a$. A graph is *apex-free ℓ-almost-embeddable* if it is $(g,p,k,0)$-almost-embeddable for some $\ell \geqslant g, p, k$.

Theorem 2.1 ([19]). — *For every positive integer t, there exists an integer ℓ such that every K_t-minor-free graph has a tree-decomposition of adhesion at most ℓ such that each torso is ℓ-almost-embeddable.*

For every positive integer n, let A_n denote the *apex $n \times n$ grid*; that is, the graph obtained from the $n \times n$ grid by adding a universal vertex. The next theorem concerns the structure of apex-minor-free graphs. The statement is implied by a characterization of apex-minor-free graphs [9, Theorem 25, (6) $\Rightarrow$ (5)].

Theorem 2.2 ([9]). — *For every positive integer ℓ, there exists some integer n such that every graph that has a tree-decomposition of adhesion at most ℓ where each torso is apex-free ℓ-almost-embeddable is A_n-minor-free.*

Finally, we will need the notion of *clique-sum*. Let k be a positive integer, $C_1 = \{v_1, \dots, v_k\}$, a clique in a graph G_1, $C_2 = \{w_1, \dots, w_k\}$, a clique in a graph G_2. A *k-clique-sum* of G_1 and G_2 is any graph G obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each $i \in [k]$ and then possibly deleting some edges in C_1 $(= C_2)$.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 first for graphs of girth 5. We then explain how the construction can be generalized so that the result holds for arbitrarily large girth.

THEOREM 3.1. — *For every positive integer t, there is a $K_6^{(1)}$-induced-minor-free graph G of girth 5 such that G is not in RIG$(\{H : H$ is K_t-minor-free$\})$.*

We fix any positive integer t. By Theorem 2.1, there exists some integer $\ell := \ell(t)$ such that every K_t-minor-free graph has a tree-decomposition of adhesion at most ℓ where each torso is ℓ-almost-embeddable. By Theorem 2.2, there exists some integer $n := n(\ell)$ such that every graph that has a tree-decomposition of adhesion at most ℓ where each torso is apex-free ℓ-almost-embeddable is A_n-minor-free. We may assume that $n \geqslant \ell + 1$. We now construct our graph G.

Construction of G

Since the $n \times n$ grid is K_5-minor-free, the apex $n \times n$ grid A_n is K_6-minor-free. Let B_n be $nA_n^{(1)}$, that is, the disjoint union of n copies of the 1-subdivision of A_n, also equal to the 1-subdivision of the disjoint union of n copies of A_n. We now set a total order $\prec$ of $V(B_n)$, and a traceable (i.e., admitting a Hamiltonian path) spanning supergraph B'_n of B_n, whose Hamiltonian path defines the successor relation of $\prec$.

The vertices of each copy of $A_n^{(1)}$ appear consecutively along $\prec$. The graph B'_n is obtained by adding to each copy of $A_n^{(1)}$ the red edges of Figure 3.1. Note that this includes an edge between the top-left vertex of the grid and the apex of the previous copy of $A_n^{(1)}$ (leftmost vertex in the figure). The order $\prec$ within each $A_n^{(1)}$ is given by the Hamiltonian path in blue, starting at the top-left vertex of the grid to the apex. Like B_n, the graph B'_n is also K_6-minor-free. The graph B'_n is not part of the construction and we will only use it in the proof of Claim 3.3.

To finish the construction, we add to B_n the disjoint union of n paths $P_1, \ldots, P_n$ of length $2|V(B_n)| - 1$, and make for every $i \in [|V(B_n)|]$ and $j \in [n]$, the $(2i-1)$-st vertex of P_j, denoted by $p_{j,i}$, adjacent to the i-th vertex of B_n along $\prec$, denoted by b_i. Call G the resulting graph. As a side note, if we replaced each copy of $A_n^{(1)}$ in G by K_1, then the graph obtained is a Pohoata–Davies Grid (see Figure 1.1).

The following three lemmas prove Theorem 3.1.

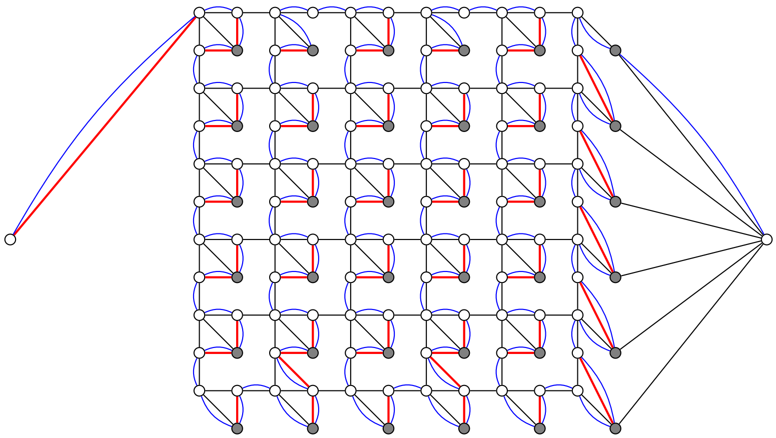

Figure 3.1. The graphs B_n, B'_n and the order $\prec$. We only represented one entire copy of $A_n^{(1)}$. Black edges represent B_n. Together with the red edges, they form B'_n. Every vertex filled in gray is adjacent to the apex vertex to the right (we only drew some of these edges for legibility). The Hamiltonian path of B'_n in blue defines the successor relation of $\prec$.

Lemma 3.2. — *G has girth at least 5.*

Proof. — B_n is the 1-subdivision of a simple graph, hence has girth at least 6. $G - V(B_n)$ is a disjoint union of paths, thus does not contain any cycle. Any cycle going through $V(G) \setminus V(B_n)$ has at least two consecutive edges within $G - V(B_n)$. We conclude as no distinct pair of vertices within the same connected component of $V(G) \setminus V(B_n)$ shares a neighbor in $V(B_n)$. □

The proof of the next lemma does not use the exact nature of B_n and its augmentation B'_n with a Hamiltonian path, only that B'_n is K_6-minor-free, and that the adjacencies to the disjoint paths $P_1, P_2, \ldots$ follow this Hamiltonian path.

Lemma 3.3. — *G is $K_6^{(1)}$-induced-minor-free.*

Proof. — Assume for the sake of contradiction that G admits $K_6^{(1)}$ as an induced minor. We will then build a minor model of K_6 in B'_n, which, we know, does not exist.

Let $\mathcal{M}$ be an induced minor model of $K_6^{(1)}$ in G such that

- every branch set of a subdivision vertex of $K_6^{(1)}$ is a singleton,

- if such a singleton is on some P_j and its two neighbors on P_j are in the two adjacent branch sets (one in each), then the singleton cannot be a vertex $p_{j,i}$ (it has to be a degree-2 vertex in between some $p_{j,i}$ and $p_{j,i+1}$), and
- each branch set is inclusion-wise minimal.

It is easy to see that this can always be done. Let $X_1, \ldots, X_6 \in \mathcal{M}$ be the branch sets corresponding to the branching vertices of $K_6^{(1)}$. We denote by $\{s_{k,k'}\}$ the branch set (of the subdivision vertex) adjacent to X_k and $X_{k'}$, for $k \neq k' \in [6]$. For each $k \in [6]$, let

$$Y_k := (X_k \cap V(B'_n)) \cup \left\{ b_i \;:\; \begin{array}{l} \exists\, j \in [n],\ p_{j,i} \in X_k \text{ and} \\ \nexists\, k' < k \in [6],\ j' \in [n],\ p_{j',i} \in X_{k'} \end{array} \right\},$$
$$\text{and} \quad Y'_k := Y_k \cup \{s_{k,k'} \in V(B'_n) \setminus (Y_1 \cup \cdots \cup Y_6) \,:\, k < k'\}.$$

We now show that $Y'_1, \ldots, Y'_6$ is a minor model of K_6 in B'_n.

Claim 1. — *The sets $Y'_1, \ldots, Y'_6$ are pairwise disjoint.*

Proof of Claim. — Suppose there exists some $b_i \in Y'_k \cap Y'_{k'}$ with $k < k'$. From the definition of $Y_1, \ldots, Y_6$ and $Y'_1, \ldots, Y'_6$, it should be that $b_i \in X_k$ and $p_{j,i} \in X_{k'}$ for some $j \in [n]$ or that $b_i \in X_{k'}$ and $p_{j,i} \in X_k$ for some $j \in [n]$. But that would make X_k and $X_{k'}$ adjacent. ♢

To further show that the sets $Y'_1, \ldots, Y'_6$ are connected and pairwise adjacent in B'_n, we need the following notion and claims. An *interval* I of some X_k is a subset of consecutive positive integers such that there is a connected component J of $G[X_k \cap V(P_j)]$ for some $j \in [n]$ such that $\{i \,:\, p_{j,i} \in V(J)\} = I$.

Claim 2. — *For any $k \neq k' \in [6]$, any interval I of X_k, and any interval I' of $X_{k'}$, it cannot be that $I \subseteq I'$ (and symmetrically $I' \subseteq I$). Furthermore, at most one vertex of $\{b_i \,:\, i \in I \cap I'\}$ can be in a branch set of $\mathcal{M}$, namely $s_{k,k'}$.*

Proof of Claim. — If $I \subseteq I'$, then X_k is a subpath of P_j for some $j \in [n]$, as otherwise X_k and $X_{k'}$ would be adjacent. But then X_k has at most two neighbors that are not neighbors of $X_{k'}$, a contradiction to realize the 4 branch sets adjacent to X_k but not to $X_{k'}$. The rest of the claim follows because $\{s_{k,k'}\}$ is the only branch set adjacent to both X_k and $X_{k'}$, and X_k and $X_{k'}$ are non-adjacent. ♢

We can extend a bit the previous claim.

Claim 3. — *For any pairwise distinct $k, k', k'' \in [6]$, any interval I of X_k, any interval I' of $X_{k'}$, and any interval I'' of $X_{k''}$, it cannot be that*

$I \subseteq I' \cup I''$. Furthermore, if $s_{k',k''} = p_{j,i}$ for some $j \in [n]$, it cannot be that $I \subseteq I' \cup I'' \cup \{i\}$.

Proof of Claim. — Again, any such inclusion would imply that X_k is a subpath of some P_j. But then X_k has at most two neighbors that are not neighbors of $X_{k'} \cup X_{k''}$ ($\cup \{s_{k',k''}\}$), a contradiction to realize the 3 branch sets adjacent to X_k but not to $X_{k'}$ nor $X_{k''}$. ♢

As $\mathcal{M}$ is minimal, Claims 2 and 3 imply in particular that there is at most one pair I, I' of intervals of $X_k, X_{k'}$ with $I \cap I' \neq \emptyset$, per $k \neq k' \in [6]$. As another direct consequence of Claims 2 and 3, we get the following.

Claim 4. — *For any pairwise distinct $k, k', k'' \in [6]$, any interval I of X_k and any interval I' of $X_{k'}$ such that $I \cap I' \neq \emptyset$ and $\min(I) < \min(I')$, there is no $i \in [\min(I') - 1, \max(I) + 1]$ such that $p_{j,i} \in X_{k''}$ for some $j \in [n]$ (or $b_i \in X_{k''}$).*

The next two claims complete the proof.

Claim 5. — *The sets $Y'_1, \ldots, Y'_6$ are connected in B'_n.*

Proof of Claim. — For any $k \in [6]$, and any $u, v \in Y'_k$, we exhibit a u–v path P in B'_n such that $V(P) \subseteq Y'_k$. (As we do not need to show that P is a path, we call it so, but only argue that it is a walk, which is sufficient.) Let $u' \in V(G)$ (resp. $v' \in V(G)$) be the vertex in $(V(G) \setminus V(B'_n)) \cap X_k$ causing that $u \in Y'_k$ (resp $v \in Y'_k$) if this applies, or $u' := u$ (resp. $v' := v$), otherwise. Let P' be a u'–v' path in G such that $V(P') \setminus \{u', v'\} \subseteq X_k$. Observe that u' and v' may be equal to some $s_{k,k'}$ with $k < k'$, and thus not be in X_k themselves. In which case, we simply run the following arguments with their neighbors in P' (which are in X_k). Hence, we may as well suppose that $u', v' \in X_k$.

If P' is a subpath of some P_j, we have $u' = p_{j,i}$ and $v' = p_{j,i'}$, no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i}$ or $p_{j',i'}$, and no other $X_{k'}$ contains $b_{i''}$ for any i'' between i and i'. By Claim 2, it means that for any integer i'' between i and i', no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i''}$, and no other $X_{k'}$ contains $b_{i''}$. In particular, all such vertices $b_{i''}$ are in Y'_k, and this makes the path P between u and v.

More generally, the path P' alternates between maximal subpaths contained in $V(G) \setminus V(B_n)$ and maximal subpaths contained in $V(B_n)$. The latter are kept to build P. We then mimic each maximal subpath contained in $V(G) \setminus V(B_n)$ with a path of B'_n included in Y'_k, with the appropriate endpoints. By Claim 2, in P', every maximal subpath $p_{j,i} \ldots p_{j,i'}$ in

$V(G) \setminus V(B_n)$ surrounded by two subpaths in $V(B_n)$ is such that the corresponding vertices $b_i \dots b_{i'}$ are all in Y'_k, hence form the desired subpath of P in B'_n.

We finally move to the case when P' starts with a subpath $u' = p_{j,i} \dots p_{j,i'} \neq v'$ maximal in $V(G) \setminus V(B_n)$; the case when P' ends with such a maximal subpath is dealt with symmetrically. We know that $b_{i'} \in X_k$, no $X_{k'}$ with $k' < k$ contains some vertex $p_{j',i}$, and no other $X_{k'}$ contains some vertex $b_{i''}$ where i'' is between i and i'. Thus by Claim 2, all the vertices $b_i \dots b_{i'}$ are in Y'_k, the desired subpath of P in B'_n. ♢

Claim 6. — *The sets $Y'_1, \dots, Y'_6$ are pairwise adjacent in B'_n.*

Proof of Claim. — For any $k \neq k' \in [6]$, let $u \in X_k, u' \in X_{k'}$ be such that $us_{k,k'}, u's_{k,k'} \in E(G)$.

Assume first that $s_{k,k'} = b_i$ for some $i \in [|V(B'_n)|]$. If at most one $\ell \in \{k, k'\}$ (thus, at most one $\ell \in [6]$) is such that $p_{j,i} \in X_\ell$ for some $j \in [n]$, then either $s_{k,k'} \in Y'_k$ and $u' \in V(B'_n)$, or $s_{k,k'} \in Y'_{k'}$ and $u \in V(B'_n)$; so Y'_k and $Y'_{k'}$ are adjacent in B'_n. If, instead, there are j, j' such that $p_{j,i} \in X_k$ and $p_{j',i} \in X_{k'}$, consider the intervals I, I' of $X_k, X_{k'}$ associated to $p_{j,i}, p_{j',i}$. Claim 4 implies that there is some i' such that $b_{i'} \in Y'_k$ and $b_{i'+1} \in Y'_{k'}$; so, again, Y'_k and $Y'_{k'}$ are adjacent in B'_n.

We next assume that $s_{k,k'} \in V(G) \setminus V(B'_n)$.

First consider the case both u and u' are also in $V(G) \setminus V(B'_n)$. Let I, I' be their associated interval, and assume without loss of generality that $\max(I) < \min(I')$. By the second item of the conditions satisfied by $\mathcal{M}$, $\min(I') - \max(I) = 1$. By Claim 3, there is no $k'' \in [6] \setminus \{k, k'\}$ such that $X_{k''}$ contains some vertex $p_{j,i}$ or b_i with $i \in [\max(I), \min(I')]$. Besides, X_k (resp. $X_{k'}$) contains no vertex $p_{j,\min(I')}$ nor $b_{\min(I')}$ (resp. $p_{j,\max(I)}$ nor $b_{\max(I)}$). Therefore, $b_{\max(I)} \in Y'_k$ and $b_{\min(I')} = b_{\max(I)+1} \in Y'_{k'}$, thus Y'_k and $Y'_{k'}$ are adjacent in B'_n.

Finally consider, without loss of generality, that $s_{k,k'} = p_{j,i}$, $p_{j,i-1} \in X_k$, and $b_i \in X_{k'}$. By Claim 2, there is no $\ell \in [6] \setminus \{k\}$ such that X_ℓ contains some vertex $p_{j',i-1}$ nor b_{i-1}. Thus $b_{i-1} \in Y'_k$. As $b_i \in Y'_{k'}$, we have that Y'_k and $Y'_{k'}$ are adjacent. ♢

Claims 1, 5 and 6 imply that $Y'_1, \dots, Y'_6$ is a K_6 minor model in B'_n; a contradiction. □

Lemma 3.4. — *For every K_t-minor-free graph H, G is not a region intersection graph over H.*

Proof. — Suppose, for contradiction, that there is a K_t-minor-free graph H for which $G \in \mathrm{RIG}(H)$. Let $\mathcal{R} = (R_v \subseteq H : v \in V(G))$ be a collection of connected subgraphs of H such that $uv \in E(G)$ if and only if $V(R_u) \cap V(R_v) \neq \emptyset$. By Theorem 2.1, H has a tree-decomposition $\mathcal{T} = (W_x : x \in V(T))$ of adhesion at most ℓ where each torso is ℓ-almost-embeddable.

We claim that there is an $x \in V(T)$ such that the bag W_x intersects $V(R_v)$ for each vertex $v \in V(B_n)$. For each vertex $v \in V(B_n)$, the set $\{x \in V(T) : V(R_v) \cap W_x \neq \emptyset\}$ is a subtree of T. By the Helly property for subtrees, it suffices to show that any two such subtrees meet.

Assume, for contradiction, that there exist $u, v \in V(B_n)$ such that $V(R_u)$ and $V(R_v)$ do not intersect a common bag. Since $\mathcal{T}$ has adhesion at most ℓ, there is a set $S \subseteq V(H)$ with $|S| \leqslant \ell$ whose deletion separates $V(R_u)$ and $V(R_v)$. By construction, G contains n u–v paths $uQ_1v, \dots, uQ_nv$ with $Q_i \subseteq P_i$. So, for each $i \in [n]$, the connected subgraph $Q_i^\star = \bigcup(R_p : p \in V(Q_i))$ of H connects R_u to R_v, hence meets S. Since $Q_1, \dots, Q_n$ are pairwise anti-complete, the subgraphs $Q_1^\star, \dots, Q_n^\star$ are pairwise vertex-disjoint, forcing $|S| \geqslant n \geqslant \ell + 1$, a contradiction.

Therefore, there is a bag W_x in $\mathcal{T}$ intersecting all regions R_v for $v \in V(B_n)$. Since every adhesion set is a clique in a torso, $V(R_v) \cap W_x$ induces a connected subgraph R'_v in $H\langle W_x \rangle$ for every $v \in V(B_n)$. However, there may be an edge $uv \in E(B_n)$ for which $V(R'_u) \cap V(R'_v) = \emptyset$. Nevertheless, since $V(R_u) \cap V(R_v) \neq \emptyset$, there is an adhesion set $S = W_x \cap W_y$ (for some edge $xy \in E(T)$) such that $V(R'_u) \cap S \neq \emptyset$ and $V(R'_v) \cap S \neq \emptyset$. Choose vertices $a \in V(R'_u) \cap S$ and $b \in V(R'_v) \cap S$. Then $ab \in E(H\langle W_x \rangle)$. Add a vertex w to $H\langle W_x \rangle$ adjacent to both a and b then include w in the connected subgraphs R'_u and R'_v. Repeating this procedure for every such edge produces a supergraph H' of $H\langle W_x \rangle$ built by performing 2-clique-sums with triangles together with a collection $(R'_v \subseteq H' : v \in V(B_n))$ of connected subgraphs in H' that realizes B_n as a region intersection graph over H'.

Let $Z \subseteq W_x$ be the set of apex vertices in $H\langle W_x \rangle$. Since $|Z| \leqslant \ell$ and B_n consists of $n \geqslant \ell+1$ anti-complete copies of $A_n^{(1)}$, there exists a copy of $A_n^{(1)}$, denoted as $\widetilde{A}_n^{(1)}$, for which $\bigcup(V(R_x) : x \in V(\widetilde{A}_n^{(1)})) \cap Z = \emptyset$. Let $\widetilde{H}$ be the subgraph of H' induced by $\bigcup(V(R_x) : x \in V(\widetilde{A}_n^{(1)}))$. Then $V(\widetilde{H}) \cap Z = \emptyset$. As such, $\widetilde{H}$ has a tree-decomposition with adhesion at most 2 where one torso is an apex-free ℓ-almost embeddable graph and the other torsos are triangles. By Theorem 2.2, $\widetilde{H}$ is A_n-minor-free. However, since $\widetilde{A}_n^{(1)} \in \mathrm{RIG}(\widetilde{H})$ and $\widetilde{A}_n^{(1)}$ is isomorphic to $A_n^{(1)}$, Lemma 1.2 implies A_n is a minor of $\widetilde{H}$, giving us the desired contradiction. □

We now explain how to modify the construction in Theorem 3.1 to force the girth to be arbitrarily large. Fix positive integer g. Define $B_{g,n}$ to be

$nA_n^{(g)}$, that is the disjoint union of n copies of the g-subdivision of A_n. Then $B_{g,n}$ has girth $3(g+1)$. We define a total order $\prec$ of $V(B_{g,n})$ by using the same strategy of that given by Figure 3.1. Similar to before, we add to $B_{g,n}$ the disjoint union of n paths $P_1, \ldots, P_n$ of length $g|V(B_{g,n})|-1$ and make, for every $i \in [|V(B_{g,n})|]$ and $j \in [n]$, the $(gi-1)$-st vertex of P_j adjacent to the i-th vertex of $B_{g,n}$ along $\prec$. Call the resulting graph $G_{g,n}$. Since $G_{g,n} - B_{g,n}$ is a disjoint union of paths, it does not contain any cycle. Any cycle going through $V(G_{g,n}) \setminus V(B_{g,n})$ has at least $g-1$ consecutive edges within $G_{g,n} - V(B_{g,n})$. Since no pair of vertices within the same connected component of $V(G_{g,n}) \setminus V(B_{g,n})$ shares a neighbor in $V(B_{g,n})$, we conclude that every cycle in $G_{g,n}$ has length at least g. Since Claims 3.3 and 3.4 also generalize to $G_{g,n}$, this completes the proof of Theorem 1.3.

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BIBLIOGRAPHY

[1] T. Abrishami, M. Briański, J. Davies, X. Du, J. Masaříková, P. Rzążewski & B. Walczak, Burling graphs in graphs with large chromatic number, 2025, https://arxiv.org/abs/2510.19650.

[2] N. Alon, P. D. Seymour & R. Thomas, A Separator Theorem for Graphs with an Excluded Minor and its Applications, in *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing, May 13–17, 1990, Baltimore, Maryland, USA* (H. Ortiz, ed.), ACM Press, 1990, p. 293–299, https://doi.org/10.1145/100216.100254.

[3] M. Bonamy, É. Bonnet, H. Déprés, L. Esperet, C. Geniet, C. Hilaire, S. Thomassé & A. Wesolek, Sparse graphs with bounded induced cycle packing number have logarithmic treewidth, *J. Comb. Theory, Ser. B* **167** (2024), p. 215–249, https://doi.org/10.1016/j.jctb.2024.03.003.

[4] É. Bonnet, J. Hodor, T. Korhonen & T. Masařík, Treewidth is Polynomial in Maximum Degree on Weakly Sparse Graphs Excluding a Planar Induced Minor, 2023, https://arxiv.org/abs/2312.07962.

[5] R. Campbell, J. Davies, M. Distel, B. Frederickson, J. P. Gollin, K. Hendrey, R. Hickingbotham, S. Wiederrecht, D. R. Wood & L. Yepremyan, Treewidth, Hadwiger Number, and Induced Minors, 2024, https://arxiv.org/abs/2410.19295.

[6] C. Dallard, M. Milanic & K. Storgel, Treewidth versus clique number. III. Tree-independence number of graphs with a forbidden structure, *J. Comb. Theory, Ser. B* **167** (2024), p. 338–391, https://doi.org/10.1016/J.JCTB.2024.03.005.

[7] J. Davies, Oberwolfach report 1/2022, https://doi.org/10.4171/OWR/2022/1, 2022.

[8] J. Davies, R. Hickingbotham, F. Illingworth & R. McCarty, Fat minors cannot be thinned (by quasi-isometries), 2024, https://arxiv.org/abs/2405.09383.

[9] V. Dujmović, P. Morin & D. R. Wood, Layered separators in minor-closed graph classes with applications, *J. Comb. Theory, Ser. B* **127** (2017), p. 111–147, https://doi.org/10.1016/j.jctb.2017.05.006.

[10] P. Gartland, Quasi-Polynomial Time Techniques for Independent Set and Beyond in Hereditary Graph Classes, PhD Thesis, UC Santa Barbara, 2023.

[11] A. Georgakopoulos & P. Papasoglu, Graph minors and metric spaces, *Combinatorica* **45** (2025), article no. 33 (29 pages), https://doi.org/10.1007/s00493-025-00150-6.

[12] T. Korhonen & D. Lokshtanov, Induced-Minor-Free Graphs: Separator Theorem, Subexponential Algorithms, and Improved Hardness of Recognition, in *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA 2024, Alexandria, VA, USA, January 7–10, 2024* (D. P. Woodruff, ed.), Society for Industrial and Applied Mathematics, 2024, p. 5249–5275, https://doi.org/10.1137/1.9781611977912.188.

[13] J. R. Lee, Separators in region intersection graphs, in *Proceedings of the 8th Innovations in Theoretical Computer Science Conference* (C. H. Papadimitriou, ed.), LIPIcs – Leibniz International Proceedings in Informatics, vol. 67, Schloss Dagstuhl, 2017, Article 1, 8 p., https://doi.org/10.4230/LIPIcs.ITCS.2017.1.

[14] R. J. Lipton & R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979), no. 2, p. 177–189, https://doi.org/10.1137/0136016.

[15] D. Lokshtanov, Personal communication, 2025.

[16] R. McCarty, Structurally Sparse Graphs, Lecture series at the Structural Graph Theory Bootcamp, Warsaw, https://sites.google.com/view/strug/main, 2023.

[17] A. C. Pohoata, Unavoidable induced subgraphs of large graphs, Senior theses, Department of Mathematics, Princeton University, 2014.

[18] N. Robertson & P. D. Seymour, Graph Minors I–XXIII, *J. Combin. Theory, Ser. B & J. Algorithms* (1983–2012).

[19] ———, Graph minors. XVI. Excluding a non-planar graph, *J. Comb. Theory, Ser. B* **89** (2003), no. 1, p. 43–76, https://doi.org/10.1016/S0095-8956(03)00042-X.

[20] F. W. Sinden, Topology of thin film RC circuits, *Bell Syst. Tech. J.* **45** (1966), no. 9, p. 1639–1662.

[21] S. Wiederrecht, Graph searching in RIGs, in *Open Problems, GRASTA 2023 Workshop*, 2023, Section 1.3, Problem 3: https://www-sop.inria.fr/teams/coati/events/grasta2023/slides/Grasta23_OpenProblems.pdf.

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