Innov. Graph Theory **1**, 2024, pp. 21–32
https://doi.org/10.5802/igt.2

INNOVATIONS IN GRAPH THEORY

THE OVERFULL NINE DRAGON TREE CONJECTURE IS TRUE

by Sebastian MIES & Benjamin R. MOORE (*)

Abstract. — Chen, Kim, Kostochka, West, and Zhu conjectured a strengthening of the Nine Dragon Tree Theorem that every graph that is (k, d)-sparse and has no overfull set decomposes into $k+1$ forests such that one of the forests has maximum degree d. Intuitively, the conjecture says that while the fractional arboricity bound in the Nine Dragon Tree Theorem is tight, we can relax the bound very slightly and still get the same result, so long as the obvious obstruction does not occur - that we can still decompose into $k+1$ forests. We prove this conjecture.

1. Introduction

In this paper, graphs may have parallel edges, but no loops. Also, we use the notation that $e(H) = |E(H)|$ and $v(H) = |V(H)|$. Recall that a *decomposition* is a partitioning of the edge set of a graph into subgraphs. This paper will be interested in decomposing graphs into forests. Since a forest on n vertices has at most $n-1$ edges, it is easy to see that if a graph G decomposes into k forests, $e(G) \leqslant k(v(G)-1)$, and further that this inequality holds over every subgraph. Rather surprisingly, Nash-Williams' Theorem [10] states that the obvious necessary condition is sufficient. That is, that a graph G decomposes into k forests if and only if $\gamma(G) \leqslant k$, where

$$\gamma(G) = \max_{H \subseteq G, v(H) \geqslant 2} \frac{e(H)}{v(H)-1}.$$

We call $\gamma(G)$ the *fractional arborcity of G*. While the theorem gives an exact characterization, one might hope to strengthen the theorem in certain instances. In particular, we can observe that the parameter γ is not

Keywords: Nine Dragon Tree, Fractional arboricity, (k, d)-sparse.
2020 *Mathematics Subject Classification:* 05C70.
(*) Benjamin Moore is funded by the European Union (ERC, RANDSTRUCT, project number 101076777) and thanks the support.

always an integer. Thus, if we have graphs G_1, G_2, G_3 with $\gamma(G_1) = 2.5$, $\gamma(G_2) = 2.1$, and $\gamma(G_3) = 2.99$, then Nash-Williams' Theorem says that all of these graphs decompose into three forests, but cannot decompose into two forests. But of course, one should anticipate the graph with fractional arboricity 2.1 to be much sparser than the graph with fractional arboricity 2.99, and so one might anticipate more structure could be deduced for the sparser graph.

While there are many possible ways to gain additional structure on the forest decomposition, for this paper we interested in the following question: what fractional arboricity do we need to ensure we can find a forest decomposition where one of the forests has maximum degree d?

Hopefully this question looks natural and interesting in its own right, but if one needs more convincing, we give some of the original motivation. At one point there was a large line of research trying to figure out the so-called game chromatic number of planar graphs. We briefly describe the problem: We have two players Alice and Bob and k colours. Starting with Alice, Alice and Bob alternatively each pick a vertex and colour it. Alice wins the game if the graph ends up properly k-coloured (i.e. no edge has its endpoints coloured the same colour), and Bob wins otherwise. Rather surprisingly, Zhu showed there is a relationship between forest decompositions with bounded maximum degree and the game chromatic number:

Theorem 1.1 ([12]). — *Suppose G decomposes into two forests, F_1, F_2 such that F_2 has maximum degree d. Then the game chromatic number of G is at most $4 + d$.*

Thus, this theorem motivates finding forest decompositions with bounded maximum degree. Montassier, Ossona de Mendez, Raspaud, and Zhu [9] posited the Nine Dragon Tree Conjecture[(1)] (now theorem) as an answer to this question:

Theorem 1.2 (Nine Dragon Tree Theorem [4]). — *Let G be a graph. Let k and d be positive integers. If $\gamma(G) \leqslant k + \frac{d}{k+d+1}$, then there is a decomposition into $k+1$ forests, where one of the forests has maximum degree at most d.*

Many partial results were given (see [11, 9, 5, 1]) before Jiang and Yang gave a beautiful proof of the entire conjecture.

Further, the fractional arboricity bound given in the Nine Dragon Tree Theorem is best possible in the following sense:

(1) named after a tree in Kaohsiung, Taiwan which is far from acyclic

Theorem 1.3 ([9]). — *For any positive integers k and d there are arbitrarily large simple graphs G and a set $S = \{e_1, \ldots, e_{d+1}\} \subseteq E(G)$ of $d+1$ edges such that $\gamma(G - S) = k + \frac{d}{k+d+1}$ and G does not decompose into $k+1$ forests where one of the forests has maximum degree d.*

Thus, it looks like the story is essentially complete. However, two possible generalizations were proposed. The more famous one is the so-called Strong Nine Dragon Tree Conjecture:

Conjecture 1.4 (Strong Nine Dragon Tree Conjecture [9]). — *Let G be a graph. Let k and d be positive integers. If $\gamma(G) \leqslant k + \frac{d}{k+d+1}$, then there is a decomposition into $k+1$ forests, where one of the forests has every connected component having at most d edges.*

This conjecture has attracted some amount of attention - see for example [8, 9, 11, 5, 6], and still remains fairly wide open. However, there is a different way to generalize the Nine Dragon Tree Theorem, which is the focus of this paper. The high level idea is simple, it says that while the fractional arboricity bound is best possible, if we forbid small dense subgraphs, we should be able to strengthen the bound slightly. It requires definitions to state.

Definition 1.5. — *For integers k and d, we say a graph G is (k,d)-*sparse *if for every subgraph H of G we have*

$$\beta(H) := (k+1)(k+d)v(H) - (k+d+1)e(H) - k^2 \geqslant 0$$

We pause to give some intuitive explanation of this parameter via comparison to the fractional arboricity of a graph. A graph G has fractional arboricity at most $k + \frac{d}{k+d+1}$ if and only if for every subgraph H of G we have

$$(k+1)(k+d)v(H) - (k+d+1)e(H) - k^2 - kd - k - d \geqslant 0.$$

Thus being (k,d)-sparse is very nearly the same thing as fractional arboricity being less than $k + \frac{d}{k+d+1}$, except we have this small $-kd-k-d$ additive term difference. Thus more graphs are (k,d)-sparse than have fractional arboricity at most $k + \frac{d}{k+d+1}$. Of course, this leaves open the possibility that a graph is (k,d)-sparse, but does not even decompose into $k+1$ forests by Nash-Williams' Theorem. Thus, this motivates the following definition:

Definition 1.6. — *Fix a positive integer k. For a graph G a subgraph H is overfull if*

$$e(H) > (k+1)(v(H) - 1).$$

Our result is the following, confirming a conjecture in [1]:

Theorem 1.7. — *Every graph which is (k,d)-sparse and has no overfull subgraph decomposes into $k+1$ forests such that one of the forests has maximum degree d.*

Observe that this implies the Nine Dragon Tree Theorem since every graph with $\gamma(G) \leqslant k + \frac{d}{k+d+1}$ is also (k,d)-sparse. Further, such a graph has no overfull set, as the existence of an overfull set implies the fractional arboricity is strictly larger than $k+1$. Prior to our theorem, Theorem 1.7 was only known when $k \leqslant 2$ and $d \neq 1$ [1].

Naturally, one would ask if we can strengthen the condition (k,d)-sparse even further. However, a construction in [5] shows we cannot:

Theorem 1.8 ([5]). — *There exist arbitrarily large graphs G with no overfull set and such that for all induced subgraphs $H \neq G$, we have $\beta(H) \geqslant 0$, but $\beta(G) = -1$, and G does not decompose into $k+1$ forests such that one of the forests has maximum degree d.*

Note this is best possible, as β is an integer. It seems increasingly likely that the Strong Nine Dragon Tree Conjecture is true, and further that it can be strengthened to an overfull version. We make the natural conjecture:

Conjecture 1.9. — *Every graph which is (k,d)-sparse and has no overfull set decomposes into $k+1$ forests such that one of the forests has every component containing at most d edges.*

As this conjecture implies the Strong Nine Dragon Tree Conjecture, it will likely be hard to solve. Nevertheless, the current approaches towards the Strong Nine Dragon Tree Conjecture, if they lead to a full resolution of the conjecture, appear as if they will extend to a proof of Conjecture 1.9.

Now, let us make some remarks about the proof of Theorem 1.7. One might note that (k,d)-sparsity is not the most natural of definitions. While it happens to be best possible from Theorem 1.8, this is not the original motivation. The original motivation is coming from how the original papers attempted to prove the Nine Dragon Tree Theorem. They wanted to use the so called *potential method* (and in fact, created it), and using (k,d)-sparseness allows a more easy facilitation of the technique. It is a slightly strange phenomenon, but due to the inductive nature of the potential method, it can be easier to prove stronger statements due to the leverage gained by the inductive hypothesis. See for example [2] for a comprehensive overview of the potential method. Thus, one might anticipate that to prove Theorem 1.7 we would give a potential method proof of the

Nine Dragon Tree Theorem. This is surprisingly not the case, and we actually proceed in a similar fashion to the only known proof of Nine Dragon Tree Theorem, and also all of the Strong Nine Dragon Tree Theorem papers (see [8, 7, 4, 3, 6, 11])

Our proof proceeds as follows. We start by observing a vertex minimal counterexample decomposes into $k + 1$ forests where k of the forests are spanning trees. This is a standard lemma proven in [4], we will omit its proof. With this, our goal is to take some forest decomposition (which we pick carefully) $T_1, \ldots, T_k, F$ where T_i are spanning trees and modify it to a new decomposition where F satisfies the theorem. To do this, we will pick a vertex $v \in V(F)$ which has too large degree, orient all edges of $T_1, \ldots, T_k$ towards v and "explore" from it to create what we call the "exploration subgraph". For those knowledgeable with the Strong Nine Dragon Tree proofs, this subgraph is exactly the same as the exploration subgraphs in those papers, except now we are rooted at a specific large degree vertex, rather than a component that is too large. For those unfamiliar, the exploration graph is the induced subgraph of all vertices reachable from v by directed paths, where we view the edges of F as bidirected. Now, our goal is to argue either we can reduce the degree of v in F without creating other large degree vertices, or argue that the exploration subgraphs fractional arboricity is too large. So we proceed by the standard arguments from the Nine Dragon Tree Theorem paper that shows two vertices joined by a directed edge in some T_i have the sum of their degrees being at least d, and that components of F cannot have too many "small" components near them. The new idea to deduce the overfull conjecture is rather simple, we show that in the component of F containing v, we can exploit the fact that all edges are oriented towards v to gain even more structure over the "nearby components" with few edges (henceforth called a small child). In particular, we will show that one cannot "generate" a small child if you are a neighbour of v in the same component of F as v. This turns out to be sufficient to prove that the exploration graph is not (k, d)-sparse, assuming we could not make any local exchange, completing the theorem. As the paper heavily relies on techniques from previous papers, we recommend reading either [4] or [3] first.

The structure of the paper is as follows. In Section 2, we define our minimal counterexample. In Section 3 we prove our key lemma on children generated by neighbours of the root vertex, as well as recall lemmas proven in [4]. Then in Section 4 we observe that a density calculation proves the theorem.

2. Setting up the minimal counterexample

The goal of this section is to set up everything we need to define our minimal counterexample. Note that as we will also consider digraphs, we will use the notation (u, v) is a directed edge from u to v. For the rest of the paper we fix integers $k, d \in \mathbb{N}$, and always assume that we have a graph G which is a counterexample with minimum number of vertices to Theorem 1.7. The first observation we need is that G decomposes into k spanning trees and another forest. First we make the following obvious observation:

Observation 2.1. — *If G has no overfull set, then G has fractional arboricity at most $k+1$, and thus decomposes into $k+1$ forests.*

Proof. — If G has fractional arboricity strictly larger than $k+1$, then there exists an induced subgraph H such that $e(H) > (k+1)(v(H)-1)$, and thus H is overfull, a contradiction. □

To strengthen this to get that G decomposes into k spanning trees and another forest now follows from a minor tweak to the proof to Lemma 2.1 of [4] so we omit the proof:

Lemma 2.2 ([4]). — *Every graph G that is a vertex minimal counterexample to Theorem 1.7 admits a decomposition into forests $T_1, \dots, T_k, F$ such that $T_1, \dots, T_k$ are spanning trees.*

Given a decomposition of G, we will want to measure how close it is to satisfying Theorem 1.7.

Definition 2.3. — *The* residue function $\rho(F)$ *of a forest F is defined as*

$$\sum_{v \in V(F)} \max\{0, \deg(v) - d\}.$$

We will want to find the decomposition with one forest minimizing the residue function.

Notation 2.4. — Over all decompositions into k spanning trees, $T_1, \dots, T_k$, and a forest F we choose one where F minimizes ρ. This forest F has a component R^* with a vertex of degree at least $d+1$, as G is a counterexample. We choose a vertex $r \in V(R^*)$ of maximum degree (which is at least 2) in R^*. We fix R^* and r for the rest of the paper.

Definition 2.5. — *We define $\mathcal{F}$ to be the set of decompositions into forests $(T_1, \dots, T_k, F)$ of G such that $T_1, \dots, T_k$ are spanning trees of G; R^* is a connected component of the undirected forest F and the edges of*

$T_1, \dots T_k$ are directed towards r. We let $\mathcal{F}^* \subseteq \mathcal{F}$ be the set of decompositions $(T_1, \dots, T_k, F) \in \mathcal{F}$ such that $\rho(F) = \rho^*$.

The next definition is simply to make it easier to talk about decompositions in $\mathcal{F}$.

Definition 2.6. — *Let $\mathcal{T} = (T_1, \dots, T_k, F) \in \mathcal{F}$. We say that the (directed) edges of $T_1, \dots, T_k$ are* blue edges *and the (undirected) edges of F are* red edges*. We define $E(\mathcal{T}) := E(T_1) \cup \dots \cup E(T_k) \cup E(F)$. For a subgraph $H \subseteq (V(G),\ E(\mathcal{T}))$ we write $E_b(H)$ and $E_r(H)$ for the set of blue and red edges of H, respectively. Furthermore, we write $e_r(H) = |E_r(H)|$.*

For a decomposition $(T_1, \dots, T_k, F) \in \mathcal{F}$, and $T \in \{T_1, \dots, T_k, F\}$, we write $\deg_T(v)$ to mean the degree of v in T. In the case of the red forest F, we will abuse notation and let $\deg_F(v) = \deg(v)$. Finally, we can define the critical subgraph which we will focus on for the rest of the paper:

Definition 2.7. — *Let $\mathcal{T} \in \mathcal{F}$. The* exploration subgraph *$H_{\mathcal{T}}$ of $\mathcal{T}$ is the subgraph of $(V(G),\ E(\mathcal{T}))$, where the vertex set $V(H_{\mathcal{T}})$ consists of all vertices v for which there is a sequence of vertices $r = x_1, \dots, x_l = v$ such that for all $1 \leqslant i < l$ it holds: $(x_i, x_{i+1}) \in E_b(\mathcal{T})$ or $x_i x_{i+1} \in E_r(\mathcal{T})$, and the set of edges of $H_{\mathcal{T}}$ is defined as*

$$E(H_{\mathcal{T}}) = \big\{xy \in E_r(\mathcal{T}) \,\big|\, x, y \in V(H_{\mathcal{T}})\big\} \cup \big\{(x, y) \in E_b(\mathcal{T}) \,\big|\, x, y \in V(H_{\mathcal{T}})\big\}.$$

We also call a connected component of $(V(G), E_r(\mathcal{T}))$ a red component.

Now we turn our focus to the notion of legal orders, which is an ordering of components of F that loosely tells us in what order we should augment the decomposition.

Definition 2.8. — *Let $(T_1, \dots T_k, F) = \mathcal{T} \in \mathcal{F}$. Let $\sigma = (R_1, \dots, R_t)$ be a sequence of all red components in $H_{\mathcal{T}}$. We say σ is a* legal order *for $\mathcal{T}$ if $R_1 = R^*$, and further for each $1 < j \leqslant t$, there is an $i_j < j$ such that there is a blue directed edge (x_j, y_j) with $x_j \in V(R_{i_j})$ and $y_j \in V(R_j)$.*

It will be useful to compare legal orders, and we will do so using the lexicographic ordering.

Definition 2.9. — *Let $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$. Suppose $\sigma = (R_1, \dots, R_t)$ and $\sigma' = (R'_1, \dots, R'_{t'})$ are legal orders for $\mathcal{T}$ and $\mathcal{T}'$, respectively. We say σ is* smaller than *σ', denoted $\sigma < \sigma'$, if $(e(R_1), \dots, e(R_t))$ is lexicographically smaller than $(e(R'_1), \dots, e(R'_{t'}))$. If $t \neq t'$, we extend the shorter sequence with zeros in order to make the orders comparable.*

To make it easier to discuss legal orders, we introduce some more vocabulary:

Definition 2.10. — *Suppose* $\sigma = (R_1, \ldots, R_t)$ *is a legal order for* $\mathcal{T} \in \mathcal{F}$*. We say that* R_i *is a* parent *of* R_j *with respect to* σ*, if* $i < j$ *holds and if there is a blue edge* (x, y) *with* $x \in V(R_i), y \in V(R_j)$*. In this case we also call* R_j *a child of* R_i *with respect to* σ *that is generated by the edge* (x, y)*.*

Note that in the above definition, a component may have many parents, and further aside from R^*, all red components have a parent. Further, a child may be generated by many blue edges.

Definition 2.11. — *Let* $(T_1, \ldots, T_k, F) = \mathcal{T} \in \mathcal{F}$ *and let* $\sigma = (R_1, \ldots, R_t)$ *be a legal order for* $\mathcal{T}$*. Compliant to Definition 2.8 we choose a blue edge* (x_j, y_j) *for all* $1 < j \leqslant t$*. There might be multiple possibilities for this, but we simply fix one choice for* σ*. We then denote* $T_\sigma := (V(H_{\mathcal{T}}),\ E_r(H_{\mathcal{T}}) \cup \{(x_j, y_j) \mid 1 < j \leqslant t\})$*, which defines a tree that we call the* auxiliary tree *of* σ*. We always consider* T_σ *to be rooted at* r*.*

With this tree in mind, each red component has a unique parent with the exception of R^*, which has no parent.

Now we are in position to define our counterexample. As already outlined, G is a vertex-minimal counterexample to the theorem. Further, we pick a legal order σ^* for a decomposition $\mathcal{T}^* = (T_1^*, \ldots, T_k^*, F^*) \in \mathcal{F}^*$ such that there is no legal order σ with $\sigma < \sigma^*$ for any $\mathcal{T}' \in \mathcal{F}^*$. We will use these notations for the minimal legal order and decomposition throughout the rest of the paper.

3. Structure of the exploration subgraph

Under the above set up, in [4] they show that the red components of the exploration subgraph are well behaved. In particular, they prove the following:

Lemma 3.1 (Corollary 2.5 from [4]). — *Let* C *be a child of* K *with respect to* σ^* *that is generated by* (x, y)*. If* $\deg(y) < d$*, then* $\deg(x) \geqslant d$*.*

Lemma 3.2 (Lemma 2.6 from [4]). — *Let* K *be a red component, and suppose* C_1 *and* C_2 *are children generated by* (x, x') *and* (y, y') *respectively. If* $xy \in E(K)$*, then either* $\deg(x') \geqslant d$ *or* $\deg(y') \geqslant d$*.*

We only need one more lemma:

Lemma 3.3. — *Let C be a child of R^* with respect to σ^* generated by $(x,y) \in E(T)$ for some $T \in \{T_1^*, \dots, T_k^*\}$. If $\deg(y) \leqslant d-1$, then $x \neq r$, and $xr \notin E(R^*)$.*

Proof. — We have that $x \neq r$ as r has no outgoing edges in T. If $xr \in E(R^*)$, then as r is the only sink in T, $T' := T + (x,r) - (x,y)$ is a tree. Without loss of generality, suppose that $T = T_1^*$, and consider the forest decomposition $(T', T_2^*, \dots, T_k^*, F^* - xr + xy)$. As y has degree at most $d-1$, it now has degree at most d in $F^* - xr + xy$, the degree of x stays the same, and the degree of r decreases by one. But this means the residue function decreased, a contradiction. □

4. The density calculation

In this section we show that our exploration graph is not (k,d)-sparse. We start off by defining an important set and subgraph. Let $\mathcal{K}$ be the set of red components of $H_{\mathcal{T}}$ with a vertex of degree at least d.

Definition 4.1. — *In an arbitrary fashion we assign each red component C that is not R^* to a vertex x such that there is a blue arc $(x,x') \in E_b(\mathcal{T}^*)$ generating C. Let $\mathcal{C}(x)$ denote the set of components that have been assigned to $x \in V(H_{\mathcal{T}^*})$. Furthermore, for $H \subseteq H_{\mathcal{T}^*}$ let $\mathcal{C}(H) := \mathcal{C}(V(H)) := \bigcup_{x \in X} \mathcal{C}(x)$ and*

$$H_{\mathcal{C}} := \left(V(H) \cup \bigcup_{C \in \mathcal{C}(H)} V(C),\ E(H) \cup \bigcup_{C \in \mathcal{C}(H)} E(C) \right).$$

Now observe the following partitioning of the red subgraph of the exploration subgraph that is justified by Lemma 3.1:

Claim 4.2. — $E_r(H_{\mathcal{T}}) = \dot{\bigcup}_{K \in \mathcal{K}} E(K_{\mathcal{C}})$.

Proof. — If a component C of $H_{\mathcal{T}}$ does not have a vertex of degree at least d, then it is not the root and hence has a parent K. If K does not contain a vertex of degree at least d, then we contradict Lemma 3.1 and thus C is contained in $K_{\mathcal{C}}$. □

Let us make some observations about the density of $K_{\mathcal{C}}$.

Lemma 4.3 (Claim 2.8 from [4]). — *Let K be a red component and K' be a connected subgraph of K. Suppose K' satisfies the following conditions:*

(1) If $\deg_{K'}(x) < d$, then $\mathcal{C}(x) = \varnothing$.

(2) There is a vertex $x \in V(K')$ with $\deg_{K'}(x) \geqslant d$.

Then

$$\frac{e(K'_{\mathcal{C}})}{v(K'_{\mathcal{C}})} \geqslant \frac{d}{d+k+1}.$$

Corollary 4.4. — *Let $K \in \mathcal{K} - R^*$. Then $\frac{e(K_{\mathcal{C}})}{v(K_{\mathcal{C}})} \geqslant \frac{d}{d+k+1}$.*

Proof. — Apply Lemma 4.3 with $K' = K$. (2) holds for K since $K \in \mathcal{K}$. (1) holds by Lemma 3.1. □

Lemma 4.5. — *There are non-negative integers $\ell, n \in \mathbb{N}$ and disjoint subgraphs $K^1, \dots, K^n$ of R^* such that $\ell \geqslant d+1$, $\frac{e(K^i_{\mathcal{C}})}{v(K^i_{\mathcal{C}})} \geqslant \frac{d}{d+k+1}$, $e(R^*_{\mathcal{C}}) = \ell + \sum_{i=1}^{n} e(K^i_{\mathcal{C}})$ and $v(R^*_{\mathcal{C}}) = \ell + 1 + \sum_{i=1}^{n} v(K^i_{\mathcal{C}}) - n$.*

Proof. — Let $n \in \mathbb{N}$ such that $K^1, \dots, K^n$ are the components of $R^* - r$ containing a vertex of degree at least d. We want to apply Lemma 4.3. Let $i \in \{1, \dots, n\}$ and let r' be the neighbour of r in K^i. Note that (2) holds for K^i by definition. Furthermore, (1) holds for R^*, $\deg_{K^i}(x) = \deg_{R^*}(x)$ for every vertex $x \in V(K^i) - r'$ and $\mathcal{C}(r') = \varnothing$ by Lemma 3.3. Thus, $\frac{e(K^i_{\mathcal{C}})}{v(K^i_{\mathcal{C}})} \geqslant \frac{d}{d+k+1}$.

Next, let R' be the subgraph of R^* induced by r and its neighbours, as well as any vertex not in K^i for any $i \in \{1, \dots, \ell\}$. then R' is connected and $\mathcal{C}(R') = \varnothing$. Furthermore, $e(R') \geqslant d+1$ since $\deg_{R^*}(r) \geqslant d+1$. The lemma follows by letting $\ell := e(R')$. □

With this we are ready to prove the conjecture.

Lemma 4.6. — *The graph $H := H_{\mathcal{T}^*}$ satisfies*

$$(k+1)(k+d)v(H) - (k+d+1)e(H) - k^2 < 0,$$

and thus, G is not (k,d)-sparse.

Proof. — Suppose to the contrary that $\beta(H) \geqslant 0$. As $T^*_1, \dots, T^*_k$ are spanning trees, we have $e(H) = k(v(H)-1) + e_r(H)$. Thus, $\beta(H) = dv(H) + (d+k+1)(k - e_r(H)) - k^2 \geqslant 0$, which we can rearrange to

$$\frac{d}{d+k+1} \geqslant \frac{e_r(H) - k + \frac{k^2}{d+k+1}}{v(H)}.$$

Using Claim 4.2 and Lemma 4.5 we can rearrange this further to:

$$\begin{aligned}
\frac{d}{d+k+1} &\geqslant \frac{e(R^*_{\mathcal{C}}) + \sum_{K\in\mathcal{K}-R^*} e(K_{\mathcal{C}}) - k(1-\frac{k}{d+k+1})}{v(R^*_{\mathcal{C}}) + \sum_{K\in\mathcal{K}-R^*} v(K_{\mathcal{C}})} \\
&= \frac{\ell - k(\frac{d+1}{d+k+1}) + \sum_{i=1}^{n} e(K^i_{\mathcal{C}}) + \sum_{K\in\mathcal{K}-R^*} e(K_{\mathcal{C}})}{\ell + 1 - n + \sum_{i=1}^{n} v(K^i_{\mathcal{C}}) + \sum_{K\in\mathcal{K}-R^*} v(K_{\mathcal{C}})} \\
&\geqslant \frac{d+1-k(\frac{d+1}{d+k+1})}{d+2} \\
&= \frac{(d+1)\frac{d+1}{d+k+1}}{d+2} \\
&> \frac{(d+1)\frac{d}{d+1}}{d+k+1} \\
&= \frac{d}{d+k+1},
\end{aligned}$$

which is a contradiction. Thus, $\beta(H) < 0$. □

Acknowledgments

The first author thanks the Institute of Science and Technology Austria and Matthew Kwan for hosting him while this work was completed.

BIBLIOGRAPHY

[1] M. Chen, S.-J. Kim, A. V. Kostochka, D. B. West & X. Zhu, Decomposition of sparse graphs into forests: The Nine Dragon Tree Conjecture for $k \leqslant 2$, *Journal of Combinatorial Theory, Series B* **122** (2017), p. 741 – 756, https://dx.doi.org/10.1016/j.jctb.2016.09.004.

[2] D. W. Cranston, Graph coloring methods, 2024, ISBN: 979-8-218-46242-0.

[3] L. Grout & B. Moore, The pseudoforest analogue for the Strong Nine Dragon Tree Conjecture is true, *Journal of Combinatorial Theory, Series B* **145** (2020), p. 433–449, https://dx.doi.org/10.1016/j.jctb.2020.07.001.

[4] H. Jiang & D. Yang, Decomposing a Graph into Forests: The Nine Dragon Tree Conjecture is True, *Combinatorica* **37** (2017), p. 1125–1137, https://dx.doi.org/10.1007/s00493-016-3390-1.

[5] S.-J. Kim, A. V. Kostochka, D. B. West, H. Wu & X. Zhu, Decomposition of Sparse Graphs into Forests and a Graph with Bounded Degree, *Journal of Graph Theory* **74** (2013), no. 4, p. 369–391, https://dx.doi.org/10.1002/jgt.21711.

[6] S. Mies & B. Moore, The Strong Nine Dragon Tree Conjecture is True for $d \leqslant 2(k+1)$, 2024, https://arxiv.org/abs/2403.05178.

[7] S. Mies, B. Moore & E. Smith-Roberge, Beyond the Pseudoforest Strong Nine Dragon Tree Theorem, 2023, https://arxiv.org/abs/2310.00931.

[8] S. Mies & B. R. Moore, The Strong Nine Dragon Tree Conjecture is true for $d \leqslant k+1$, *Combinatorica* **43** (2022), p. 1215–1239, https://dx.doi.org/10.1007/s00493-023-00058-z.

[9] M. Montassier, P. Ossona de Mendez, A. Raspaud & X. Zhu, Decomposing a graph into forests, *Journal of Combinatorial Theory, Series B* **102** (2012), no. 1, p. 38–52, https://dx.doi.org/10.1016/j.jctb.2011.04.001.

[10] C. S. J. A. Nash-Williams, Decomposition of Finite Graphs Into Forests, *Journal of the London Mathematical Society* **39** (1964), p. 12, https://dx.doi.org/10.1112/jlms/s1-39.1.12.

[11] D. Yang, Decomposing a graph into forests and a matching, *Journal of Combinatorial Theory, Series B* **131** (2018), p. 40 – 54, https://dx.doi.org/10.1016/j.jctb.2018.01.005.

[12] X. Zhu, Refined activation strategy for the marking game, *Journal of Combinatorial Theory, Series B* **98** (2008), no. 1, p. 1–18, https://dx.doi.org/10.1016/j.jctb.2007.04.004.

Manuscript received 23rd November 2023,
accepted 19th June 2024.

Sebastian MIES
Johannes Gutenberg University
Institute of Computer Science
Mainz, Germany
smies@students.uni-mainz.de

Benjamin R. MOORE
Institute of Science and Technology
Combinatorics and Probablity Group
Klosterneuburg Austria
Benjamin.Moore@ist.ac.at

Innovations in Graph Theory is a member of the Mersenne Center for Open Publishing
ISSN: pending