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INNOVATIONS IN GRAPH THEORY

FORCING QUASIRANDOMNESS IN A REGULAR TOURNAMENT

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Abstract. — A tournament H is said to *force quasirandomness* if it has the property that a sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments of increasing orders is quasirandom if and only if the homomorphism density of H in T_n tends to $(1/2)^{\binom{v(H)}{2}}$ as $n \to \infty$. It was recently shown that there is only one non-transitive tournament with this property. This is in contrast to the analogous problem for graphs, where there are numerous graphs that are known to force quasirandomness and the well known Forcing Conjecture suggests that there are many more. To obtain a richer family of characterizations of quasirandomness in tournaments, we propose a variant in which the tournaments $(T_n)_{n\in\mathbb{N}}$ are assumed to be "nearly regular." We characterize the tournaments on at most 5 vertices which force quasirandomness under this stronger assumption.

1. Introduction

A combinatorial structure is said to be "quasirandom" if it shares various properties which hold with probability tending to 1 in a large random structure of the same kind. For example, a dense graph is quasirandom if, for any two "large enough" sets of vertices, the density of edges between them closely approximates the global edge density. Many seemingly different but formally equivalent characterizations of quasirandom graphs exist; see [10, 42, 45, 46]. For example, an equivalent definition of quasirandomness in dense graphs is that the number of closed walks of length four, i.e. homomorphisms from a 4-cycle, approximately matches

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the expectation in a random graph with the same number of vertices and edges. The well-known Forcing Conjecture of Conlon, Fox and Sudakov [11], based on a question of Skokan and Thoma [44], asserts that quasirandomness in graphs of a given edge density can be characterized in terms of the number of homomorphisms of any fixed bipartite graph with at least one cycle. The notion of quasirandomness was first introduced by Rödl [42], Thomason [45, 46] and Chung, Graham and Wilson [10]. Since then, it has continued to attract a great deal of attention [5, 7, 8, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 26, 30, 31, 32, 33, 44] and was an inspiration behind combinatorial limit theory [6, 36, 37].

Our focus in this paper is on obtaining new characterizations of quasirandomness in tournaments, as studied in [2, 9, 16, 17, 24, 27, 28]; recall that a *tournament* is a directed graph (i.e. a *digraph*) with no loops and exactly one arc between every pair of distinct vertices. A *homomorphism* from a digraph H to a digraph D is a map $f : V(H) \to V(D)$ such that $f(u)f(v) \in A(D)$ whenever $uv \in A(H)$, where $A(F)$ denotes the set of arcs of a digraph F. Let $\hom(H, D)$ denote the number of homomorphisms from H to D and define the *homomorphism density* of H in D to be

$$t(H, D) := \frac{\hom(H, D)}{v(D)^{v(H)}},$$

where $v(F)$ denotes the number of vertices in a digraph F. Intuitively, $t(H, D)$ is the probability that a uniformly random function of the form $f : V(H) \to V(D)$ is a homomorphism.

If T_n is a uniformly random tournament on n vertices, i.e. each arc is directed randomly in one of the two possible directions with equal probability independent of all other arcs, and H is a tournament on k vertices, then

$$\mathbb{E}(t(H, T_n)) = (1 - o(1)) \left(\frac{1}{2}\right)^{\binom{k}{2}},$$

where the asymptotics are as $n \to \infty$. Furthermore, an application of the Azuma–Hoeffding Inequality (see [1, Theorem 7.2.1]) tells us that the random variable $t(H, T_n)$ is tightly concentrated around this expected value. This motivates the following definition. A sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments is said to be *quasirandom* if, for every $k \geqslant 1$ and every k-vertex tournament H,

$$\lim_{n\to\infty} t(H, T_n) = \left(\frac{1}{2}\right)^{\binom{k}{2}}.$$

In other words, a sequence of tournaments is quasirandom if every finite "pattern" appears with nearly the same frequency in a tournament in this

sequence as it would in a sequence of uniformly random tournaments. Note that every quasirandom sequence of tournaments must satisfy $v(T_n) \to \infty$ as $t(H, T_n) = 0$ if H is a tournament with $v(H) > v(T_n)$. An important idea in quasirandomness is that it can often be characterized in terms of the frequencies of only a small number of patterns. The following definition is studied in, e.g., [2, 17, 27].

DEFINITION 1.1. — *A tournament H with k vertices is said to* force quasirandomness *if any sequence $(T_n)_{n\in\mathbb{N}}$ with $v(T_n) \to \infty$ and $\lim_{n\to\infty} t(H, T_n) = (1/2)^{\binom{k}{2}}$ is quasirandom.*

Loosely speaking, this means that if the density of H in T_n is asymptotically equal to its density in a random tournament, then this is true for any other tournament H' as well. In contrast to many combinatorial structures, such as graphs, the class of tournaments which force quasirandomness is very restricted. The *transitive tournament* on k vertices, denoted TT_k, is the tournament obtained by listing the vertices in a row and directing all arcs to the right. The fact that TT_k forces quasirandomness for all $k \geqslant 4$ is a slight extension of [35, Exercise 10.44], and was reproved in [17] using the flag algebra method. Coregliano, Parente and Sato [16] also used flag algebras to obtain an example of a 5-vertex non-transitive tournament that forces quasirandomness. This is tournament H_{17} in the list of all tournaments on at most 5 vertices provided in Figure 1.1. Bucić, Long, Shapira and Sudakov [2] proved that there are no non-transitive tournaments on 7 or more vertices which force quasirandomness. Finally, Hancock et al. [27] showed that there are no tournaments on at most 6 vertices which force quasirandomness, apart from transitive tournaments on $4, 5$ or 6 vertices and H_{17}, completing the characterization of tournaments which force quasirandomness.

THEOREM 1.2 (See [2, 16, 17, 27, 35]). — *A tournament H forces quasirandomness if and only if it is a transitive tournament on at least 4 vertices or is isomorphic to the tournament H_{17} in Figure 1.1.*

Our focus in this paper is on tournaments H which force quasirandomness under the additional assumption that the sequence $(T_n)_{n\in\mathbb{N}}$ is *nearly regular* meaning that, for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that, for all $n \geqslant n_0(\varepsilon)$, all but at most $\varepsilon \cdot v(T_n)$ vertices of T_n have out-degree between $(1/2 - \varepsilon)\, v(T_n)$ and $(1/2 + \varepsilon)\, v(T_n)$. Note that any nearly regular sequence of tournaments must satisfy $v(T_n) \to \infty$ since it is impossible for all vertices of T_n to have out-degree greater than $\frac{v(T_n)-1}{2}$. Also, a sequence of uniformly random tournaments with the number of vertices tending to

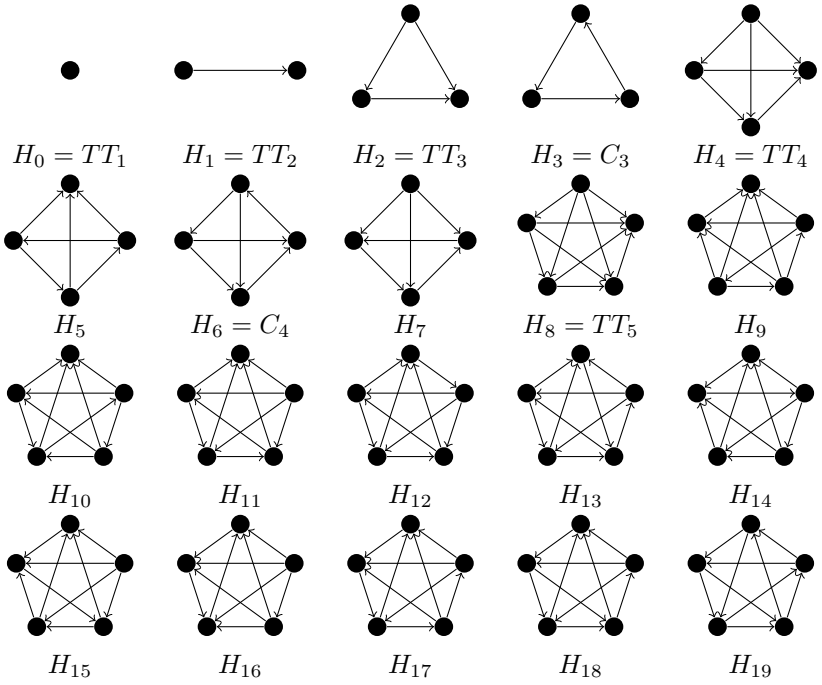


Figure 1.1. The tournaments on at most 5 *vertices, up to isomorphism. There are 20 such tournaments labelled* $H_0, \dots, H_{19}$*. The tournaments* H_0, H_1, H_2, H_4 *and* H_8 *are transitive. The tournament* H_3 *is the cyclic tournament on 3 vertices and is also referred to as* C_3*. Similarly,* H_6 *is the unique* 4*-vertex tournament containing a spanning cycle and is also referred to as* C_4*.*

infinity is nearly regular with probability one; this follows from a simple application of the Chernoff bound.

Definition 1.3. — *A tournament* H *on* k *vertices is said to* force quasirandomness in regular tournaments *if any nearly regular sequence* $(T_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(H, T_n) = (1/2)^{\binom{k}{2}}$ *is quasirandom.*

Our main result is the following theorem on tournaments on at most 5 vertices which force quasirandomness in regular tournaments. In contrast to Theorem 1.2, the class of tournaments which force quasirandomness in regular tournaments seems as though it could be significantly broader than those which force quasirandomness in general (not necessarily regular) tournaments; see Questions 6.2 and 6.3. We note that a few cases of the next

theorem follow from the earlier results of [16, 35]; see Propositions 4.2, 5.1 and 5.3.

THEOREM 1.4. — *Let $H_0, \dots, H_{19}$ be the list of all tournaments on at most 5 vertices up to isomorphism, as in Figure 1.1.*

(i) The tournaments $H_0, H_1, H_2, H_3, H_9, H_{12}, H_{16}, H_{18}$ and H_{19} do not force quasirandomness in regular tournaments.

(ii) The tournaments $H_4, H_5, H_6, H_7, H_8, H_{10}, H_{11}, H_{13}, H_{14}, H_{15}$ and H_{17} force quasirandomness in regular tournaments.

The rest of the paper is organized as follows. In the next section, we translate the notions of homomorphism density, near regularity and quasirandomness into the language of "limit objects" for tournaments. Then, in Section 3, we derive some necessary conditions and sufficient conditions for the property of forcing quasirandomness in regular tournaments that will be used in our proofs. In Section 4, we present several families of tournament limit constructions which allow us to prove the assertion in Theorem 1.4 (i). Then, in Section 5, we give a brief introduction to the flag algebra method and use it to prove several inequalities on linear combinations of homomorphism densities of tournaments on 3 and 5 vertices and show that equality holds only for a quasirandom sequence of tournaments. Theorem 1.4 (ii) is then derived from these results. We conclude the paper in Section 6 by discussing a few open problems.

2. Preliminaries

We begin with some standard notation and terminology related to tournaments. The *adjacency matrix* of a tournament T with vertices $v_1, \dots, v_n$ is the $n \times n$ matrix A in which the entry on the ith row and jth column is equal to 1 if T contains an arc from v_i to v_j and 0 otherwise. The *out-degree* of a vertex $v \in V(T)$, denoted $d_T^+(v)$ is the number of arcs of T that start at v and the *in-degree* $d_T^-(v)$ is the number of arcs of T that end at v. In other words, the out-degree is the sum of the entries in the row corresponding to v in the adjacency matrix of T, and the in-degree is the sum of the entries in the column corresponding to v. Additionally, a tournament T is *regular* if every vertex of T has out-degree $\frac{v(T)-1}{2}$.

Our next goal is to introduce some tenets of the standard limit theory for tournaments; for a deeper treatment of combinatorial limits (focusing especially on graphs), see Lovász [36]. We note that, for our purposes, there is no formal mathematical advantage of dealing with limit objects

as opposed to infinite sequences of tournaments. The main benefits are stylistic in nature; i.e. the limit setting allows us to avoid distracting the reader by repeatedly mentioning that "n is large" and keeping track of floors, ceilings, and lower order asymptotic terms which are technically necessary but ultimately immaterial.

Following the terminology[(1)] in [24, 25, 27, 39, 43], a *tournamenton* is a measurable function $W : [0,1]^2 \to [0,1]$ with the property that $W(x,y) + W(y,x) = 1$ for all $x, y \in [0,1]$. Essentially, a tournamenton is just a "continuous generalization" of the adjacency matrix of a tournament; the property $W(x,y)+W(y,x) = 1$ is analogous to the fact that the adjacency matrix A of a tournament T satisfies $A_{i,j} + A_{j,i} = 1$ for all $i \neq j$. Given a digraph D with vertex set $V(D) = \{v_1, \dots, v_k\}$ and a tournamenton W, the *homomorphism density* of D in W is defined to be the k-fold integral

$$t(D,W) := \int_0^1 \cdots \int_0^1 \prod_{v_i v_j \in A(D)} W(x_i, x_j) \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

Another way of viewing $t(D,W)$ is in terms of the following random procedure. Pick k points $x_1, \dots, x_k$ from $[0,1]$ uniformly at random and independently from one another and let $\{v_1, \dots, v_k\}$ be a set of vertices. Then, for each $i \neq j$, add an arc from v_i to v_j with probability $W(x_i, x_j)$ and from v_j to v_i otherwise. The tournament T resulting from this is known as a *W-random tournament*. It is not hard to see that $t(D,W)$ is precisely the probability that all arcs of D are contained in a W-random tournament T.

A sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments with $v(T_n) \to \infty$ is said to *converge* to a tournamenton W if $\lim_{n\to\infty} t(H, T_n) = t(H, W)$ for every tournament H. The following proposition is a standard extension of a fundamental theorem in graph limits of Lovász and Szegedy [38]. It is not new; similar ideas are used in, e.g., [4, Subsection 2.2].

Proposition 2.1. — *For every sequence $(T_n)_{n\in\mathbb{N}}$ of finite tournaments with $v(T_n) \to \infty$ there is a subsequence of $(T_n)_{n\in\mathbb{N}}$ that converges to a tournamenton W. On the other hand, for every tournamenton W, if $(T_n)_{n\in\mathbb{N}}$ is a sequence of W-random tournaments with $v(T_n) \to \infty$, then $(T_n)_{n\in\mathbb{N}}$ converges to W with probability* 1.

It will sometimes be convenient to extend the notion of quasirandom forcing from tournaments to sets of formal linear combinations of tournaments. Let $\mathcal{T}$ denote the set of all finite tournaments and $\mathbb{R}[\mathcal{T}]$ be the set of

(1) It seems that the community has not settled on a definitive name for limit objects of tournaments. Other sources use *tourneyon* [48], *tournament limit* [5] or *tournament kernel* [47]; however, the term tournamenton seems like the most popular variant so far.

all formal linear combinations of tournaments; i.e. expressions of the form $\sum_{i=1}^{t} \alpha_i \cdot H_i$ for $\alpha_1, \dots, \alpha_t \in \mathbb{R}$ and $H_1, \dots, H_t \in \mathcal{T}$. We regard a single tournament H as being the same as the formal linear combination $1 \cdot H$. Given such a formal linear combination $H = \sum_{i=1}^{t} \alpha_i \cdot H_i$ and a tournament T, define $t(H,T) = \sum_{i=1}^{t} \alpha_i \cdot t(H_i,T)$ and let $t(H,1/2) = \sum_{i=1}^{t} \alpha_i \cdot (1/2)^{\binom{v(H_i)}{2}}$. For a tournamenton W, we define $t(H,W)$ similarly. We say that a set $S \subseteq \mathbb{R}[\mathcal{T}]$ *forces quasirandomness* if every sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ such that $v(T_n) \to \infty$ and $\lim_{n\to\infty} t(H,T_n) = t(H,1/2)$ for all $H \in S$ is quasirandom. We say that S *forces quasirandomness in regular tournaments* if the same conclusion holds for any sequence of nearly regular tournaments. For $H \in \mathbb{R}[\mathcal{T}]$, we say that H *forces quasirandomness* or *forces quasirandomness in regular tournaments* if the set $\{H\}$ does. The following proposition follows from standard results in combinatorial limit theory; in particular, it is a very slight extension of [27, Proposition 1].

PROPOSITION 2.2. — *Let $S \subseteq \mathbb{R}[\mathcal{T}]$. The set S forces quasirandomness if and only if every tournamenton W such that $t(H,W) = t(H,1/2)$ for all $H \in S$ has the property that $W(x,y) = 1/2$ for almost all $(x,y) \in [0,1]^2$.*

Given a tournamenton W and $x \in [0,1]$, define the *out-degree* of x to be

$$d^+_W(x) := \int_0^1 W(x,y)\mathrm{d}y$$

and the *in-degree* to be

$$d^-_W(x) := \int_0^1 W(y,x)\mathrm{d}y.$$

We say that W is *regular* if $d^+_W(x) = 1/2$ for almost every $x \in [0,1]$. Note that, in contrast to the usual in- and out-degrees in tournaments, degrees in tournamentons are always between zero and one. One should think of these quantities as being analogous to the "normalized" degrees $d^+_T(v)/v(T)$ and $d^-_T(v)/v(T)$ in a tournament T rather than the degrees themselves.

3. Basic Reductions

We now obtain a mild extension of a classical result of Kendall and Babington Smith [29] which roughly says that, among tournaments T on n vertices, the homomorphism density of TT_3 in T is minimized by a tournament which is "as regular as possible." The following result is well known.

PROPOSITION 3.1. — *Every tournamenton W satisfies $t(TT_3,W) \geqslant 1/8$ with equality if and only if W is regular.*

Proof. — Let W be any tournameton. Then

$$t(TT_3, W) = \int_0^1 \int_0^1 \int_0^1 W(x,y)W(x,z)W(y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

but also

$$t(TT_3, W) = \int_0^1 \int_0^1 \int_0^1 W(x,y)W(x,z)W(z,y)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Therefore, summing these expressions and using that $W(y,z)+W(z,y) = 1$, we get

$$\begin{aligned} 2t(TT_3, W) &= \int_0^1 \int_0^1 \int_0^1 W(x,y)W(x,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_0^1 \left(\int_0^1 W(x,y)\mathrm{d}y\right)\left(\int_0^1 W(x,z)\mathrm{d}z\right)\mathrm{d}x = \int_0^1 \left(d_W^+(x)\right)^2 \mathrm{d}x. \end{aligned} \tag{3.1}$$

Since W is a tournamenton, we have that $\int_0^1 d_W^+(x)\mathrm{d}x = 1/2$. So, by Jensen's Inequality,

$$\int_0^1 \left(d_W^+(x)\right)^2 \mathrm{d}x \geqslant \left(\int_0^1 d_W^+(x)\mathrm{d}x\right)^2 = 1/4.$$

Combining this with (3.1), we get $t(TT_3, W) \geqslant 1/8$. By strict convexity of the function $f(z) = z^2$, we have that equality holds in the above inequality if and only if $d_W^+(x) = d_W^+(y)$ for almost all x and y, which can only occur if $d_W^+(x) = 1/2$ for almost all x. □

We now derive an analogous, but opposite, result for C_3; recall from Figure 1.1 that C_3 is the cyclic tournament on 3 vertices. The following result is also well known.

Proposition 3.2. — *Every tournamenton W satisfies $t(C_3, W) \leqslant 1/8$ with equality if and only if W is regular.*

Proof. — For almost every $x \in [0,1]$, we have $d_W^+(x) + d_W^-(x) = 1$. Thus,

$$\begin{aligned} 1 &= \int_0^1 \left(d_W^+(x) + d_W^-(x)\right)^2 \mathrm{d}x \\ &= \int_0^1 \left(d_W^+(x)\right)^2 \mathrm{d}x + 2\int_0^1 d_W^+(x)d_W^-(x) + \int_0^1 \left(d_W^-(x)\right)^2 \mathrm{d}x. \end{aligned}$$

By (3.1), the first term in the rightmost expression above is $2 \cdot t(TT_3, W)$. A similar argument to the proof of (3.1) implies that the last term is $2 \cdot t(TT_3, W)$ as well, and that the middle term is $2t(C_3, W) + 2t(TT_3, W)$. Therefore, every tournamenton W satisfies

$$2t(C_3, W) + 6t(TT_3, W) = 1 \tag{3.2}$$

or, in other words,

$$t(C_3, W) = \frac{1}{2} - 3t(TT_3, W).$$

The result now follows by Proposition 3.1. □

Remark 3.3. — By following an argument similar to the proof of Proposition 3.2, but for a finite tournament T rather than a tournamenton W, one gets

$$\begin{aligned} v(T)(v(T)-1)^2 &= \sum_{v \in V(T)} \left(d_T^+(v) + d_T^-(v)\right)^2 \\ &= 2\hom(C_3, T) + 6\hom(TT_3, T) + 2\hom(TT_2, T). \end{aligned}$$

One can obtain the following analogues of the previous propositions for nearly regular sequences of tournaments by following essentially the same proofs.

PROPOSITION 3.4. — *Every sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments with $v(T_n) \to \infty$ satisfies*

$$\liminf_{n\to\infty} t(TT_3, T_n) \geqslant 1/8.$$

Moreover, $(T_n)_{n\in\mathbb{N}}$ is nearly regular if and only if $\lim_{n\to\infty} t(TT_3, T_n) = 1/8$.

PROPOSITION 3.5. — *Every sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments with $v(T_n) \to \infty$ satisfies*

$$\limsup_{n\to\infty} t(C_3, T_n) \leqslant 1/8.$$

Moreover, $(T_n)_{n\in\mathbb{N}}$ is nearly regular if and only if $\lim_{n\to\infty} t(C_3, T_n) = 1/8$.

Using the previous two propositions, we derive the following characterizations of sets of formal linear combinations of tournaments which force quasirandomness in regular tournaments.

PROPOSITION 3.6. — *Let $S \subseteq \mathbb{R}[\mathcal{T}]$. The following are equivalent:*

- *S forces quasirandomness in regular tournaments,*
- *$S \cup \{TT_3\}$ forces quasirandomness,*
- *$S \cup \{C_3\}$ forces quasirandomness.*

Proof. — We will only prove that the first statement is equivalent to the second; the analogous result with TT_3 replaced by C_3 follows from the same arguments except that we apply Proposition 3.5 instead of Proposition 3.4.

Suppose that S forces quasirandomness in regular tournaments. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of tournaments of increasing number of vertices and $\lim_{n\to\infty} t(TT_3, T_n) = 1/8$. Then $(T_n)_{n\in\mathbb{N}}$ is nearly regular by Proposition 3.4. So, if we additionally assume that $\lim_{n\to\infty} t(H, T_n) = t(H, 1/2)$

for all $H \in S$, then, since S forces quasirandomness in regular tournaments, we get that $(T_n)_{n\in\mathbb{N}}$ is quasirandom. Conversely, assume that $S \cup \{TT_3\}$ forces quasirandomness. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a nearly regular sequence of tournaments such that $\lim_{n\to\infty} t(H, T_n) = t(H, 1/2)$ for all $H \in S$. Since the sequence is nearly regular, Proposition 3.4 tells us that $\lim_{n\to\infty} t(TT_3, T_n) = 1/8$. Therefore, since $S \cup \{TT_3\}$ forces quasirandomness, we have that $(T_n)_{n\in\mathbb{N}}$ is quasirandom. □

We now get a sufficient condition for forcing quasirandomness in regular tournaments which we will use in the flag algebra proofs of Section 5.

LEMMA 3.7. — *Let H be a tournament. If there exists $\alpha, \beta \in \mathbb{R}$ such that the formal linear combination $\alpha TT_3 + \beta H$ forces quasirandomness, then H forces quasirandomness in regular tournaments. Similarly, if there exists $\alpha, \beta \in \mathbb{R}$ such that the formal linear combination $\alpha C_3 + \beta H$ forces quasirandomness, then H forces quasirandomness in regular tournaments.*

Proof. — Suppose that $\alpha TT_3 + \beta H$ forces quasirandomness. Then the set $\{TT_3, H\}$ forces quasirandomness, since any sequence $(T_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} t(TT_3, T_n) = 1/8$ and $\lim_{n\to\infty} t(H, T_n) = t(H, 1/2)$ must satisfy $\lim_{n\to\infty} t(\alpha TT_3 + \beta H, T_n) = t(\alpha TT_3 + \beta H, 1/2)$ as well. So, by Proposition 3.6, H forces quasirandomness in regular tournaments. □

We close this section with a simple necessary condition for forcing quasirandomness in regular tournaments that will be used in Section 4.

LEMMA 3.8. — *Let $H \in \mathbb{R}[\mathcal{T}]$. If there exist regular tournamentons W_0 and W_1 such that*

$$t(H, W_0) < t(H, 1/2) \quad \text{and} \quad t(H, W_1) > t(H, 1/2),$$

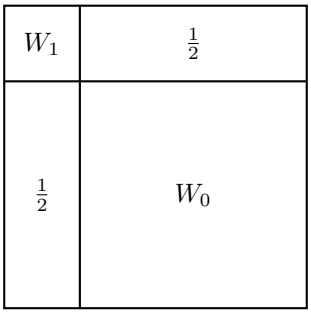
then H does not force quasirandomness in regular tournaments.

Proof. — For each $z \in (0, 1)$, let W_z be the tournamenton defined by

$$W_z(x, y) = \begin{cases} W_1(x/z, y/z) & \text{if } 0 \leqslant x, y \leqslant z, \\ W_0((x-z)/(1-z), (y-z)/(1-z)) & \text{if } z < x, y \leqslant 1, \\ 1/2 & \text{otherwise.} \end{cases}$$

An illustration of the structure of this graphon in the case $z = 1/4$ is given below. Note that, in all pictures of tournamentons, the origin of $[0, 1]^2$ is

in the top left corner, in analogy with adjacency matrices.



Then $t(H, W_z)$ is a continuous function of the variable z for $z \in [0,1]$ and satisfies $t(H, W_0) < t(H, 1/2)$ and $t(H, W_1) > t(H, 1/2)$. So, by the Intermediate Value Theorem, there must exist $z \in (0,1)$ such that $t(H, W_z) = t(H, 1/2)$. It is easily observed that W_z is regular for all $z \in (0,1)$ because W_0 and W_1 are regular.

Next, we claim that it is not the case that $W_z(x,y) = 1/2$ for almost all $(x,y) \in [0,z]^2$. Indeed, if it were the case, then, since $z > 0$, we would have that $W_1(x,y) = 1/2$ for almost all $(x,y) \in [0,1]^2$ as well. However, this would imply $t(H, W_1) = t(H, 1/2)$, which contradicts the assumption of the lemma. So, we have $W_z(x,y) \neq 1/2$ for a positive measure of points $(x,y) \in [0,z]^2$. Thus, by Proposition 2.2 and Proposition 3.2, the pair $\{H, C_3\}$ does not force quasirandomness which, by Proposition 3.6, implies that H does not force quasirandomness in regular tournaments. □

4. Negative Results

In this section, we show that $H_0, H_1, H_2, H_3, H_{12}, H_{18}$ and H_{19} do not force quasirandomness in regular tournaments. The following class of tournamentons will be useful for this purpose. Given a finite tournament T with a vertex set $\{1, \dots, k\}$, one can define a corresponding tournament W_T by dividing $[0,1]$ into intervals $I_1, \dots, I_k$ of equal measure and defining

$$W_T(x,y) := \begin{cases} 1 & \text{if } x \in I_i \times I_j \text{ for some } ij \in A(T), \\ 1/2 & \text{if } x \in I_i \times I_i \text{ for some } i \in V(T), \\ 0 & \text{otherwise.} \end{cases}$$

Observation 4.1. — *Let T be a tournament. Then T is regular if and only if W_T is regular.*

We start with the "easy cases" H_0, H_1, H_2 and H_3.

PROPOSITION 4.2. — *The tournaments H_0, H_1, H_2 and H_3 do not force quasirandomness in regular tournaments.*

Proof. — Consider any regular tournamenton W such that there is a subset of $[0,1]^2$ of positive measure on which $W \neq 1/2$. For example, one could take $W = W_T$ for any regular tournament T; e.g., $T = C_3$ would work. We have $t(H_0, W) = 1$ and $t(H_1, W) = 1/2$, as these equations hold for every tournamenton. Since $H_2 = TT_3$ and $H_3 = C_3$, we have $t(H_2, W) = t(H_3, W) = 1/8$ by Propositions 3.1 and 3.2. So, by Propositons 2.2 and 3.6, none of these tournaments force quasirandomness in regular tournaments. □

Next, we consider the tournaments H_9 and H_{16}.

PROPOSITION 4.3. — *The tournaments H_9 and H_{16} do not force quasirandomness in regular tournaments.*[(2)]

Proof. — Note that H_9 and H_{16} can be obtained from one another by reversing all arcs. Therefore, one of them forces quasirandomness in regular tournaments if and only if the other does. So, without loss of generality, we focus only on H_9. Let the five vertices of H_9 be denoted by $u_1, \dots, u_5$ taken in clockwise order starting with the top vertex in the picture of H_9 in Figure 1.1. Then u_1 is a sink, the only out-neighbour of u_2 is u_1 and the vertices u_3, u_4, u_5 form a cyclic triangle.

The tournamenton W_{C_3} is regular, and so we will be done if we can argue that $t(H_9, W_{C_3}) = 2^{-10}$. Let us compute this by hand. We take a W_{C_3}-random tournament on five vertices $v_1, \dots, v_5$ and compute the probability that v_iv_j is an arc of this random tournament if and only if u_iu_j is an arc of H_9. Let I_1, I_2 and I_3 be the three intervals used in the construction of W_{C_3}. First, if no two of the points v_3, v_4, v_5 are sampled from the same such interval, then the probability that v_1 is a sink is zero. Next, if v_3, v_4, v_5 are sampled from precisely two of the intervals I_1, I_2, I_3, then the probability that they form a cyclic triangle is zero.

Finally, suppose that v_1, v_2, v_3 are sampled from the same such interval. There are three choices of interval and, for any choice, the probability of all three of these points landing in that interval is $(1/3)^3$. Given this, the probability that they have the correct arcs among them is $(1/2)^3$. Now, the only ways in which the W_{C_3}-random tournament can have the correct arcs

(2) The example in the proof of this proposition was pointed out to us by Bernard Lidický, Daniel Kráľ, Florian Pfender and Jan Volec after posting an earlier draft of this paper to arxiv; see the acknowledgements at the end of the paper.

with non-zero probability are if (a) v_1 and v_2 are both mapped to the same interval as v_3, v_4, v_5, (b) v_2 is mapped to the same interval as v_3, v_4, v_5 and v_1 is mapped to the next interval in cyclic order, or (c) both v_1 and v_2 are mapped to the next interval in cyclic order. The probabilities of each of these events, and the probability that the W_{C_3}-random tournament has the correct arcs given these events, can be computed fairly straightforwardly to give the following:

$$t(H_9, W_{C_3}) = 3\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^1\right) = 2^{-10}$$

as desired. □

We conclude the section by using Lemma 3.8 to prove that H_{12}, H_{18} and H_{19} do not force quasirandomness in regular tournaments.

LEMMA 4.4. — *The tournaments H_{12}, H_{18} and H_{19} do not force quasirandomness in regular tournaments.*

Proof. — We will prove the lemma via three applications of Lemma 3.8. For each $z \in [0, 1/2]$, let U_z be the tournamenton defined by setting

$$U_z(x, y) = 1/2 + z(2 \cdot W_{C_3}(x, y) - 1)$$

for all $x, y \in [0, 1]$. Visually, U_z is the function obtained by dividing $[0, 1]^2$ into a regular 3×3 grid and assigning values to points in each square of the grid according to the following diagram.

$\frac{1}{2}$	$\frac{1}{2} + z$	$\frac{1}{2} - z$
$\frac{1}{2} - z$	$\frac{1}{2}$	$\frac{1}{2} + z$
$\frac{1}{2} + z$	$\frac{1}{2} - z$	$\frac{1}{2}$

Note that U_0 is nothing more than the constant function equal to $1/2$ everywhere, whereas $U_{1/2}$ is W_{C_3}. Also note that U_z is a regular tournamenton for all $z \in [0,1]$. It is tedious, but not hard, to show that

$$\begin{aligned} t(H_{12}, U_z) &= \frac{1}{1024} - \frac{1}{96}z^4 + \frac{7}{108}z^6 + \frac{1}{18}z^8, \\ t(H_{18}, U_z) &= \frac{1}{1024} - \frac{1}{96}z^4 - \frac{1}{108}z^6 - \frac{7}{162}z^8, \text{ and} \\ t(H_{19}, U_z) &= \frac{1}{1024} + \frac{5}{288}z^4 - \frac{5}{36}z^6 + \frac{5}{162}z^8. \end{aligned}$$

From this, we see that

$$\begin{aligned} t(H_{12}, U_{1/2}) &= \frac{43}{27648} > \frac{1}{1024}, \\ t(H_{12}, U_{1/3}) &= \frac{57161}{60466176} < \frac{1}{1024}, \\ t(H_{18}, U_{1/2}) &= \frac{1}{82944} < \frac{1}{1024}, \\ t(H_{19}, U_{1/3}) &= \frac{546961}{544195584} > \frac{1}{1024}, \\ t(H_{19}, U_{1/2}) &= \frac{1}{82944} < \frac{1}{1024}. \end{aligned}$$

By Lemma 3.8, we get that H_{12} and H_{19} do not force quasirandomness in regular tournaments. The last thing that we need is a tournamenton W such that $t(H_{18}, W) > \frac{1}{1024}$. Let T be the 7-vertex tournament with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Clearly, T is regular. This tournament was found by an exhaustive computer search through all tournaments on a small number of vertices. By computer, we have found that

$$t(H_{18}, W_T) \approx 0.001249886 > \frac{1}{1024}.$$

Thus, we get that H_{18} does not force quasirandomness in regular tournaments by one more application of Lemma 3.8. □

5. Positive Results

In this section, we complete the proof of Theorem 1.4 by showing that all tournaments in part (ii) of the theorem force quasirandomness in regular tournaments. We start with five examples that can be derived straightforwardly from known results.

PROPOSITION 5.1 (See [16, 35]). — *The tournaments H_4, H_5, H_7, H_8 and H_{17} force quasirandomness in regular tournaments.*

Proof. — The tournaments H_4, H_8 and H_{17} force quasirandomness by Theorem 1.2 (note that H_4 and H_8 are TT_4 and TT_5, respectively), and so they force quasirandomness in regular tournaments as well.

We consider H_5 and let $(T_n)_{n\in\mathbb{N}}$ be a nearly regular sequence of tournaments. Note that H_5 consists of a cycle on three vertices and one "sink" vertex; i.e. a vertex of out-degree zero. For each $v \in V(T_n)$, let $N_n^-(v)$ denote the in-neighbourhood of v and $d_n^-(v)$ the in-degree of v. For a set $S \subseteq V(T_n)$, let $T_n[S]$ be the subtournament of T_n induced by S. Then

$$\hom(H_5, T_n) = \sum_{v\in V(T_n)} \hom(C_3, T_n[N_n^-(v)]).$$

To see this, we note that the term corresponding to v counts homomorphisms from H_5 to T_n in which the sink is mapped to v. This can be rewritten using the fact that $\hom(C_3, T_n[N_n^-(v)]) = d_n^-(v)^3 \cdot t(C_3, T_n[N_n^-(v)])$, as follows.

$$\hom(H_5, T_n) = \sum_{v\in V(T_n)} \left(d_n^-(v)\right)^3 t(C_3, T_n[N_n^-(v)]).$$

Using Remark 3.3, we can rewrite this again as

$$\hom(H_5, T_n) = \sum_{v\in V(T_n)} \left(d_n^-(v)\right)^3 \frac{1}{2}\left(1 - 6t(TT_3, T_n[N_n^-(v)]) - o(1)\right).$$

Now, if we use the fact that $(T_n)_{n\in\mathbb{N}}$ is nearly regular, the right side of the above equality becomes

$$\begin{aligned}(v(T_n)/2)^4 - 3 \sum_{v\in V(T_n)} \left(d_n^-(v)\right)^3 t(TT_3, T_n[N_n^-(v)]) - o(v(T_n)^4)\\ = (v(T_n)/2)^4 - 3\hom(TT_4, T_n) - o(v(T_n)^4)\end{aligned}$$

where the fact that $\sum_{v\in V(T_n)} \left(d_n^-(v)\right)^3 t(TT_3, T_n[N_n^-(v)]) = \hom(TT_4, T_n)$ follows from a similar "double-counting" argument as we used before to express $\hom(H_5, T_n)$ as a summation. If we divide this by $v(T_n)^4$, we get

$$t(H_5, T_n) = (1/2)^4 - 3t(TT_4, T_n) - o(1).$$

So, if $\lim_{n\to\infty} t(H_5, T_n) = (1/2)^6$, then $\lim_{n\to\infty} t(TT_4, T_n) = (1/2)^6$. Since TT_4 forces quasirandomness (see Theorem 1.2), the sequence $(T_n)_{n\in\mathbb{N}}$ must be quasirandom, and hence we get that H_5 forces quasirandomness in regular tournaments. Note that H_7 is a cycle on three vertices with a source vertex (i.e. a vertex of in-degree zero) and so the fact that it forces quasirandomness in regular tournaments follows from a very similar argument. □

We next consider $C_4 = H_6$. The fact that H_6 forces quasirandomness in regular tournaments is easily derived from the following result of [4].[(3)]

LEMMA 5.2 (Chan, Grzesik, Král' and Noel [4, Corollary 6]). — *If W is a tournamenton and $1/2 \leqslant z \leqslant 1$ such that*

$$t(C_3, W) = \frac{1}{8}\left(z^3 + (1-z)^3\right),$$

then

$$t(C_4, W) \geqslant \frac{1}{64}\left(z^4 + (1-z)^4\right),$$

where equality holds if and only if there exists a measurable function $f : [0,1] \to [0,1/2]$ such that $W(x,y) = 1/2 + f(x) - f(y)$ for almost every $(x,y) \in [0,1]^2$.

PROPOSITION 5.3 (See [4, Corollary 6]). — *H_6 forces quasirandomness in regular tournaments.*

Proof. — Note that H_6 is C_4. Let W be a regular tournamenton. By Proposition 3.2, we have $t(C_3, W) = 1/8$. So, applying Lemma 5.2 with $z = 1$ gives us that $t(C_4, W) \geqslant 1/64$, where equality holds if and only if there is a measurable function $f : [0,1] \to [0,1/2]$ such that $W(x,y) = 1/2 + f(x) - f(y)$ for all $x, y \in [0,1]$. Since W is regular, we have, for almost all $x \in [0,1]$,

$$\frac{1}{2} = d^+_W(x) = \int_0^1 W(x,y)\mathrm{d}y = \int_0^1 \left(\frac{1}{2} + f(x) - f(y)\right)\mathrm{d}y,$$

which implies that $f(x) = \int_0^1 f(y)\mathrm{d}y$. Therefore, the function f is constant almost everywhere which tells us that $W(x,y) = 1/2$ for almost all $(x,y) \in [0,1]^2$. So, we are done by Proposition 2.2. □

The goal in the rest of this section is to prove the following five theorems which, together with Lemma 3.7 and Proposition 2.2, imply that H_{10}, H_{11}, H_{13} and H_{14} force quasirandomness in regular tournaments. Note

(3) Note that, in [4], they use C_4 to denote the 4-vertex directed cycle digraph, whereas here we let C_4 denote the unique 4-vertex tournament containing such a cycle. Because of this, the value of $t(C_4, W)$ in this paper is precisely 1/4 times the value of $t(C_4, W)$ in [4] for every tournamenton W.

that H_{15} is the tournament obtained from H_{10} by reversing all the arcs. Therefore, we will get that H_{15} forces quasirandomness in regular tournaments as well.

THEOREM 5.4. — *For every tournamenton W,*

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{10}, W) \leqslant \frac{5}{4},$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

THEOREM 5.5. — *For every tournamenton W,*

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{11}, W) \leqslant \frac{5}{4},$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

THEOREM 5.6. — *For every tournamenton W,*

$$8 \cdot t(TT_3, W) + \frac{1}{7} \cdot 1024 \cdot t(H_{13}, W) \geqslant \frac{8}{7},$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

THEOREM 5.7. — *For every tournamenton W,*

$$8 \cdot t(TT_3, W) + \frac{1}{5} \cdot 1024 \cdot t(H_{14}, W) \geqslant \frac{6}{5},$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

We now deduce our main result from these five theorems, after which we will prove the theorems themselves.

Proof of Theorem 1.4. — By Propositions 2.2, 5.1 and 5.3 and Theorems 5.4, 5.5, 5.6 and 5.7, together with the fact that and H_{15} is obtained from H_{10} by reversing all arcs, each of the tournaments H_i for $i \in \{4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 17\}$ forces quasirandomness in regular tournaments. By Proposition 4.2 and Lemma 4.4, the tournaments H_i for $i \in \{0, 1, 2, 3, 12, 18, 19\}$ do not. This completes the proof. □

We now focus our attention on the proofs of Theorems 5.4–5.7. We will apply the *flag algebra* method, introduced by Razborov [41]. This method has had a heavy impact throughout extremal combinatorics, including several successful applications to problems involving tournaments [3, 16, 17, 34]. Our presentation of the method will follow along similar lines to the treatment in [40, Section 5] for graphs.

For each tournament J and tournamenton W, the *induced density* of J in W is defined to be

$$d(J,W) := \frac{v(J)!}{\mathrm{aut}(J)} \cdot t(J,W), \tag{5.1}$$

where $\mathrm{Aut}(J)$ is the set of all automorphisms of J and $\mathrm{aut}(J) := |\mathrm{Aut}(J)|$. It is not hard to see that $d(J,W)$ is precisely the probability that a W-random tournament with $v(J)$ vertices is isomorphic to J. The following lemma just says that these probabilities sum to 1.

LEMMA 5.8. — *For any $m \geqslant 1$ and tournamenton W,*

$$\sum_{J:v(J)=m} d(J,W) = 1,$$

where the sum is over all m-vertex tournaments J up to isomorphism.

In this paper, a *flag* is a pair (F,r) where F is a tournament on vertex set $\{1,\dots,v(F)\}$ and $r \leqslant v(F)$. The vertices $\{1,\dots,r\}$ are known as the *roots* of the flag. Given a flag (F,r) and a tournamenton W, define $t_r(F,W) : [0,1]^r \to [0,1]$ by

$$t_r(F,W)(x_1,\dots,x_r) := \int_0^1 \cdots \int_0^1 \prod_{ij\in A(F)} W(x_i,x_j)\mathrm{d}x_{r+1}\cdots\mathrm{d}x_{v(F)}.$$

That is, it is the same as the usual homomorphism density, except that we only integrate over the variables x_i for $i \in \{r+1,\dots,v(F)\}$. Now, for $r \leqslant k$ and any tournament R with $V(R) = \{1,\dots,r\}$, let $\mathcal{F}_k(R)$ be the collection of all distinct (but possibly isomorphic) flags (F,r) such that F has k vertices and the subtournament of F induced by $\{1,\dots,r\}$ is equal to R (not just isomorphic to R, but has exactly the same arc set). Note that $\mathcal{F}_k(R)$ has cardinality $2^{r(k-r)+\binom{k-r}{2}}$. The following lemma uses the same idea as the proof of Proposition 3.1; that is, it exploits the fact that $W(x,y) + W(y,x) = 1$ for every tournamenton W and $(x,y) \in [0,1]^2$.

LEMMA 5.9. — *Let R be a tournament with $V(R) = \{1,\dots,r\}$ and let $k \geqslant r$. For any tournamenton W we have*

$$\sum_{(F,r)\in\mathcal{F}_k(R)} t_r(F,W)(x_1,\dots,x_r) = t_r(R,W)(x_1,\dots,x_r)$$

for almost every $(x_1,\dots,x_r) \in [0,1]^r$.

Proof. — For this proof, it is convenient to extend the notion of a flag slightly to include any pair (D,r) where D is a digraph on vertex set $\{1,\dots,v(D)\}$ and $r \leqslant v(D)$. The function $t_r(D,W)(x_1,\dots,x_r)$ is defined

in the same way as it is for tournaments. Given a tournament R on vertex set $\{1,\dots,r\}$ and a graph G with vertex set $\{1,\dots,m\}$ containing a clique on $\{1,\dots,r\}$, let $\mathcal{F}_G(R)$ be the set of all flags (D,r) where D is obtained from orienting the edges of G so that the subdigraph induced by $\{1,\dots,r\}$ is equal to R. We claim that

$$\sum_{(D,r)\in\mathcal{F}_G(R)} t_r(D,W)(x_1,\dots,x_r) = t_r(R,W)(x_1,\dots,x_r)$$

for almost every $(x_1,\dots,x_r)\in[0,1]^r$. We proceed by induction on $e(G)$. First, if every edge of G is in $\{1,\dots,r\}$, then the equality is clear, since integrating out the variables corresponding to vertices not in $\{1,\dots,r\}$ just produces factors of 1. Now, let uv be any edge of G not contained in $\{1,\dots,r\}$. For any $(D,r)\in\mathcal{F}_G(R)$ containing an arc from u to v, there is a flag $(D',r)\in\mathcal{F}_G(R)$ with all the same arcs as D except that the edge uv is oriented in the other direction. Let D'' be obtained from D by deleting the arc uv. Since $W(x,y)+W(y,x)=1$ for almost all $(x,y)\in[0,1]^2$ we have

$$t_r(D,W)(x_1,\dots,x_r)+t_r(D',W)(x_1,\dots,x_r)=t_r(D'',W)(x_1,\dots,x_r).$$

Pairing up all digraphs in $\mathcal{F}_G(R)$ in this way gives us that

$$\sum_{(D,r)\in\mathcal{F}_G(R)} t_r(D,W)(x_1,\dots,x_r) = \sum_{(D'',r)\in\mathcal{F}_{G\setminus\{e\}}(R)} t_r(D'',W)(x_1,\dots,x_r)$$

and we are done by induction. □

Given tournaments H and J with $v(H)\leqslant v(J)$, define the *injective homomorphism density* $t_{\mathrm{inj}}(H,J)$ to be the probability that a uniformly random injective function from $V(H)$ to $V(J)$ is a homomorphism. That is, it is the number of injective homomorphisms from H to J divided by the number of injective functions from $V(H)$ to $V(J)$. In the case that $v(H)>v(J)$, simply define $t_{\mathrm{inj}}(H,J)$ to be zero. Let $\mathcal{T}_{\leqslant m}$ be the set of all tournaments on at most m vertices up to isomorphism and $\mathbb{R}[\mathcal{T}_{\leqslant m}]$ be the set of all formal linear combinations of elements of $\mathcal{T}_{\leqslant m}$. We extend $t_{\mathrm{inj}}(\,\cdot\,,W)$ to elements of $\mathbb{R}[\mathcal{T}_{\leqslant m}]$ linearly, analogously to $t(\,\cdot\,,W)$. We derive the next lemma from Lemma 5.9.

LEMMA 5.10. — *For any $m\geqslant 1$, $H\in\mathbb{R}[\mathcal{T}_{\leqslant m}]$ and tournamenton W,*

$$t(H,W)=\sum_{J:v(J)=m} t_{\mathrm{inj}}(H,J)\cdot d(J,W),$$

where the sum is over all m-vertex tournaments J up to isomorphism.

Proof. — We may assume that H is a single graph, as the general result will follow from linearity. Let $r = v(H)$ and note that we may assume $V(H) = \{1, \dots, r\}$. By Lemma 5.9, we have that

$$t_r(H, W)(x_1, \dots, x_r) = \sum_{(F,r) \in \mathcal{F}_m(H)} t_r(F, W)(x_1, \dots, x_r) \tag{5.2}$$

for almost every $(x_1, \dots, x_r) \in [0,1]^r$. Integrating this over all $(x_1, \dots, x_r) \in [0,1]^r$ gives us $t(H, W)$ on the left side.

On the right side, we get $2^{r(m-r)+\binom{m-r}{2}}$ terms, each of which is of the form $t(J, W)$ for some graph J on m vertices. For each graph J on m vertices, up to isomorphism, let us compute the coefficient of $t(J, W)$. If $\hom(H, J) = 0$, then the coefficient is zero, so we assume that there is at least one injective homomorphism from H to J. So, we may assume that $V(J) = \{1, \dots, m\}$ and the subtournament of J on $\{1, \dots, r\}$ is equal to H. Consider the subgroup Γ of the symmetric group S_m consisting of bijections on $\{1, \dots, m\}$ whose restriction to $\{1, \dots, r\}$ is a homomorphism from H to J. Then $\mathrm{Aut}(J)$ is a subgroup of Γ. Now, the coefficient of $t(J, W)$ on the right side of (5.2) is equal to the number of cosets of $\mathrm{Aut}(J)$ in Γ which, by standard facts from group theory, is $\frac{\hom_{\mathrm{inj}}(H,J)(m-r)!}{\mathrm{aut}(J)}$. Putting all of this together,

$$\begin{aligned} t(H, W) &= \sum_{J : v(J) = m} \frac{\hom_{\mathrm{inj}}(H, J)(m-r)!}{\mathrm{aut}(J)} t(J, W) \\ &= \sum_{J : v(J) = m} \left(\frac{\hom_{\mathrm{inj}}(H, J)(m-r)!}{m!} \right) \cdot \left(\frac{m!}{\mathrm{aut}(J)} t(J, W) \right) \\ &= \sum_{J : v(J) = m} t_{\mathrm{inj}}(H, J) d(J, W). \end{aligned}$$

□

At this point, let us briefly pause our discussion of the flag algebra method to prove the following lemma which will be very useful in characterizing the equality cases in Theorems 5.4–5.7.

Lemma 5.11. — *If W is a tournamenton such that $8t(C_4, W) = t(TT_3, W)$ and $8t(TT_4, W) = t(TT_3, W)$, then $W = 1/2$ almost everywhere.*

Proof. — There are four tournaments on four vertices up to isomorphism. Among these, TT_4 contains four transitive triangles, C_4 contains two, and the other two tournaments, H_5 and H_7, contain three each. There are $4! = 24$ injective maps from a set of cardinality three to a set of cardinality four. So, applying Lemma 5.10 to the case $H = TT_3$ and $m = 4$ yields

$$t(TT_3, W) = \frac{4}{24} d(TT_4, W) + \frac{2}{24} d(C_4, W) + \frac{3}{24} d(H_5, W) + \frac{3}{24} d(H_7, W).$$

Now, using Lemma 5.8 to cancel out the terms involving H_5 and H_7 gives us

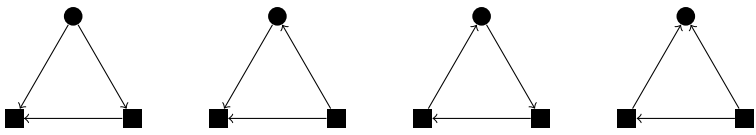
$$t(TT_3, W) - \frac{3}{24} = \frac{1}{24} d(TT_4, W) - \frac{1}{24} d(C_4, W).$$

By (5.1) and the fact that $\mathrm{aut}(TT_4) = \mathrm{aut}(C_4) = 1$, this is equivalent to

$$t(TT_3, W) - \frac{1}{8} = t(TT_4, W) - t(C_4, W).$$

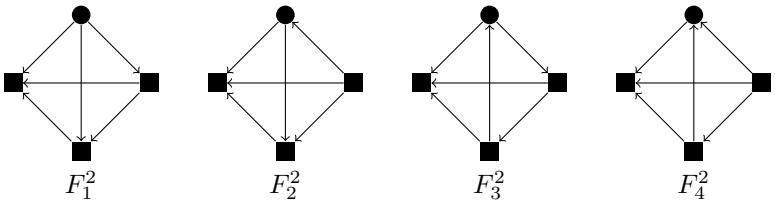
Now, using the hypothesis of the lemma, we get that the right side is zero. Therefore, we have $t(TT_3, W) = 1/8$. Applying the hypothesis once again gives us $t(TT_4, W) = 1/64$ which, by the fact that TT_4 forces quasirandomness (see Theorem 1.2), implies that $W = 1/2$ almost everywhere. □

We now present the flags that we will use in the proofs of Theorems 5.4–5.7. In each depiction of a flag (F, r), the root vertices $1, \dots, r$ are depicted as square nodes, listed left to right in increasing order. Each flag will have exactly one non-root vertex which is depicted as a circular node. We use the flags in $\mathcal{F}_3(TT_2)$, labelled by F_i^1 for $1 \leqslant i \leqslant 4$ as follows:

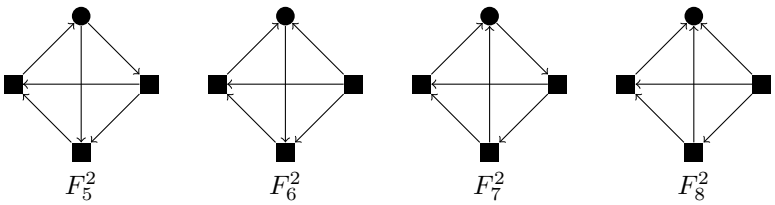


F_1^1 F_2^1 F_3^1 F_4^1

We also use the flags in $\mathcal{F}_4(TT_3)$, which we label by F_i^2 for $1 \leqslant i \leqslant 8$ as follows:

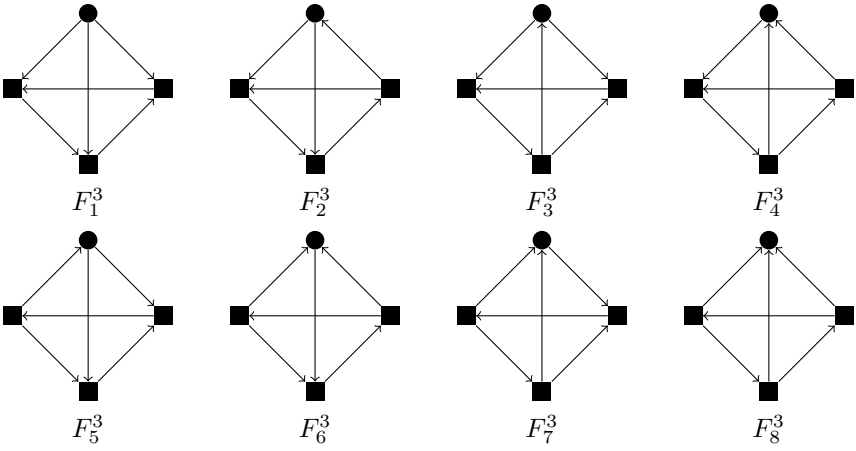


F_1^2 F_2^2 F_3^2 F_4^2



F_5^2 F_6^2 F_7^2 F_8^2

Finally, we have the elements of $\mathcal{F}_4(C_3)$, labelled F_i^3 for $1 \leqslant i \leqslant 8$ like so:



Next, we require a multiplication operation on pairs of flags.

Definition 5.12. — *Let R be a tournament with vertex set $\{1,\dots,r\}$. Say that two flags (F_1,r) and (F_2,r) are* R-compatible *if $v(F_1)=v(F_2)$ and, for each $i\in\{1,2\}$, the subtournament of F_i induced by the roots is equal to R. Say that (F_1,r) and (F_2,r) are* compatible *if they are R-compatible for some tournament R of order r.*

Definition 5.13. — *Let R be a tournament with vertex set $\{1,\dots,r\}$. Given R-compatible flags (F_1,r) and (F_2,r) and a tournamenton W, define $t_r(F_1\cdot F_2,W):[0,1]^r\to[0,1]$ by*

$$t_r(F_1\cdot F_2,W)(x_1,\dots,x_r):=\frac{t_r(F_1,W)(x_1,\dots,x_r)\cdot t_r(F_2,W)(x_1,\dots,x_r)}{t_r(R,W)(x_1,\dots,x_r)},$$

if $t_r(R,W)(x_1,\dots,x_r)\neq 0$ or $t_r(F_1\cdot F_2,W)(x_1,\dots,x_r):=0$ otherwise.

A key idea in the flag algebra method is that, for two R-compatible flags (F_1,r) and (F_2,r), the integral of $t_r(F_1\cdot F_2,W)(x_1,\dots,x_r)$ over $[0,1]^r$ can be expressed as a linear combination of $d(J,W)$ over all tournaments J with $v(F_1)+v(F_2)-v(F)$ vertices. We have actually already seen an instance of this, in disguise, in the proof of Proposition 3.1. If we let $F_1=F_2$ be the tournament on vertex set $\{1,2\}$ where 1 is the source, then $t_1(F_1\cdot F_2,W)(x_1)=\left(d_W^+(x_1)\right)^2$. Therefore, (3.1) expresses $\int_0^1 t_1(F_1\cdot F_2,W)(x_1)\mathrm{d}x_1$ as a linear combination of $d(J,W)$ over all tournaments J with 3 vertices. The same sort of "handshaking" argument applies to any other product of compatible flags to yield the following lemma.

LEMMA 5.14. — *Let R be a tournament with vertex set $\{1,\dots,r\}$, let (F_1,r) and (F_2,r) be R-compatible flags and let $m \geqslant v(F_1)+v(F_2)-v(F)$. Then there exist constants $b_r(F_1,F_2;J)$ for each tournament J on m vertices such that*

$$\int_0^1 \cdots \int_0^1 t_r(F_1\cdot F_2,W)(x_1,\dots,x_r)\mathrm{d}x_1\cdots\mathrm{d}x_r = \sum_{J:v(J)=m} b_r(F_1,F_2;J)\cdot d(J,W)$$

for every tournamenton W, where the sum on the right side is over all m-vertex tournaments up to isomorphism.

We provide an example to illustrate how to compute the coefficients $b_r(F_1,F_2;J)$ in Lemma 5.14. We also remark that all of the coefficients that are needed for our proofs are listed in Appendix A for convenience. Consider the flags F_1^1 and F_3^1 from $\mathcal{F}_3(TT_2)$ as defined earlier in this section. Then $\int_0^1\int_0^1 t(F_1^1\cdot F_3^1,W)(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2$ can be written as

$$\int_0^1\int_0^1 W(x_2,x_1)\left(\int_0^1 W(x_3,x_1)W(x_3,x_2)\mathrm{d}x_3\right)\cdot\left(\int_0^1 W(x_3,x_2)W(x_1,x_3)\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2$$

which, by a change of variables, equals to

$$\begin{aligned}&\int_0^1\int_0^1 W(x_2,x_1)\left(\int_0^1 W(x_3,x_1)W(x_3,x_2)\mathrm{d}x_3\right)\cdot\left(\int_0^1 W(x_4,x_2)W(x_1,x_4)\mathrm{d}x_4\right)\mathrm{d}x_1\mathrm{d}x_2\\ &=\int_0^1\int_0^1\int_0^1\int_0^1 W(x_2,x_1)W(x_3,x_1)W(x_3,x_2)\cdot W(x_4,x_2)W(x_1,x_4)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\\ &=\int_{[0,1]^4} W(x_2,x_1)W(x_3,x_1)W(x_3,x_2)W(x_4,x_2)W(x_1,x_4)\cdot[W(x_4,x_3)+W(x_3,x_4)]\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4.\end{aligned}$$

Expanding the sum in the above expression yields

$$\int_0^1\int_0^1 t(F_1^1\cdot F_3^1,W)(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = t(H_6,W)+t(H_7,W). \tag{5.3}$$

Now, by definition of $d(\cdot, W)$, the right side of (5.3) can be rewritten as

$$\frac{1}{24}d(H_6, W) + \frac{3}{24}d(H_7, W).$$

Thus, for these two flags, we have $b_2(F_1^1, F_3^1; H_7) = \frac{3}{24}$ and $b_2(F_1^1, F_3^1; H_6) = \frac{1}{24}$. Also, $b_2(F_1^1, F_3^1; J) = 0$ for any 4-vertex tournament not isomorphic to H_6 or H_7. For any flags (F_1, r) and (F_2, r) and tournamant J with at least $v(F_1) + v(F_2) - r$ vertices, the coefficient $b_r(F_1, F_2; J)$ in Lemma 5.14 can be computed explicitly via a similar procedure. As mentioned earlier, all of the coefficients that we require in our proofs are included in Appendix A.

Finally, we state the key lemma that will be used in the proofs of Theorems 5.4–5.7. A symmetric $t \times t$ matrix M is said to be *positive semidefinite*, written $M \succcurlyeq 0$, if all of its eigenvalues are non-negative. Equivalently, $M \succcurlyeq 0$ if and only if $\sum_{i=1}^{t}\sum_{j=1}^{t} M_{i,j}x_ix_j \geqslant 0$ for any $x_1, \ldots, x_t \in \mathbb{R}$.

Lemma 5.15. — *Let m be an integer and $H \in \mathbb{R}[\mathcal{T}_{\leqslant m}]$. Let $\ell, t_1, \ldots, t_\ell$, $r_1, \ldots, r_\ell, k_1, \ldots, k_\ell$ be positive integers such that $k_q \geqslant r_q + 1$ and $2k_q - r_q \leqslant m$ for all $1 \leqslant q \leqslant \ell$. For each $1 \leqslant q \leqslant \ell$, let $(F_1^q, r_q), \ldots, (F_{t_q}^q, r_q)$ be pairwise compatible flags on vertex set $\{1, \ldots, k_q\}$ and A_q be a positive semidefinite $t_q \times t_q$ matrix. Then every tournamenton W satisfies*

$$t(H, W) \geqslant \min_{J : v(J) = m} \left\{ t_{\mathrm{inj}}(H, J) - \sum_{q=1}^{\ell}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q} b_{r_q}(F_i^q, F_j^q; J) \cdot (A_q)_{i,j} \right\}.$$

Moreover, if W is a tournamenton such that equality holds, then, for every $1 \leqslant q \leqslant \ell$ and almost every $(x_1, \ldots, x_{r_q}) \in [0,1]^{r_q}$, the vector

$$\Big(t_{r_q}(F_1^q, W)(x_1, \ldots, x_{r_q}), \ldots, t_{r_q}(F_{t_q}^q, W)(x_1, \ldots, x_{r_q}) \Big)$$

is in the kernel of A_q.

Proof. — Let W be any tournamenton. By Lemma 5.10,

$$t(H, W) = \sum_{J : v(J) = m} t_{\mathrm{inj}}(H, J) d(J, W). \tag{5.4}$$

We claim that the following holds for any $1 \leqslant q \leqslant \ell$ and $(x_1, \ldots, x_{r_q}) \in [0,1]^{r_q}$:

$$\sum_{i=1}^{t_q}\sum_{j=1}^{t_q} (A_q)_{i,j} \cdot t(F_i^q \cdot F_j^q, W)(x_1, \ldots, x_{r_q}) \geqslant 0. \tag{5.5}$$

Indeed, if $(x_1, \ldots, x_{r_q}) \in [0,1]^{r_q}$ are such that $t(R^q, W)(x_1, \ldots, x_{r_q}) = 0$, where R^q is the subgraph of any of the flags $F_1^q, \ldots, F_{t_q}^q$ induced by $\{1, \ldots, r_q\}$, then every term of the sum is zero. On the other hand, if

$t(R^q, W)(x_1, \dots, x_{r_q}) \neq 0$, then applying the definition of $t(F_i^q \cdot F_j^q, W)$ gives us that the sum is non-negative due to the fact that A_q is positive semidefinite. Now, if we integrate out all of the variables in (5.5), we get

$$\sum_{J:v(J)=m} \sum_{q=1}^{\ell} \sum_{i=1}^{t_q} \sum_{j=1}^{t_q} (A_q)_{i,j} \cdot b_{r_q}(F_i^q, F_j^q; J) \cdot d(J, W) \geqslant 0.$$

Putting this together with (5.4) gives us

$$t(H, W) \geqslant \sum_{J:v(J)=m} \left(t_{\mathrm{inj}}(H, J) - \sum_{q=1}^{\ell} \sum_{i=1}^{t_q} \sum_{j=1}^{t_q} (A_q)_{i,j} \cdot b_{r_q}(F_i^q, F_j^q; J) \right) \cdot d(J, W).$$

Finally, by Lemma 5.8, we have that $\sum_{J:v(J)=m} d(J, W) = 1$ and so the quantity on the right side of the above inequality is bounded below by the minimum of the coefficient of $d(J, W)$ over all J. This is precisely the bound in the lemma.

Now, let us discuss the case of equality. If, for some $1 \leqslant q \leqslant \ell$, there is a positive measure subset of $[0, 1]^{r_q}$ such that the vector

$$\left(t_{r_q}(F_1^q, W)(x_1, \dots, x_{r_q}), \dots, t_{r_q}(F_{t_q}^q, W)(x_1, \dots, x_{r_q}) \right)$$

is not in the kernel of A_q, then the inequality (5.5) is strict for a positive measure of points. This, in turn, translates to a gap in the final bound. So, if equality holds, then such vectors must be in the kernel of A_q for each $1 \leqslant q \leqslant \ell$ and almost all $(x_1, \dots, x_{r_q})$. □

Without further delay, we use the flag algebra method to prove Theorems 5.4–5.7.

Proof of Theorem 5.4. — We apply Lemma 5.15 for $m = 5$, $t_2 = t_3 = 8$, $r_2 = r_3 = 3$, $k_2 = k_3 = 4$, flags F_i^2 and F_i^3 defined earlier and the following matrices A_2 and A_3:

$$A_2 = \frac{1}{245} \begin{bmatrix} 4724 & -1883 & 1081 & -4598 & 3827 & -293 & 1390 & -4248 \\ -1883 & 6512 & -4787 & 3150 & 39 & 378 & -2965 & -444 \\ 1081 & -4787 & 3856 & -2274 & -371 & -313 & 2010 & 798 \\ -4598 & 3150 & -2274 & 5490 & -3658 & 156 & -1734 & 3468 \\ 3827 & 39 & -371 & -3658 & 4600 & -420 & 945 & -4962 \\ -293 & 378 & -313 & 156 & -420 & 606 & -660 & 546 \\ 1390 & -2965 & 2010 & -1734 & 945 & -660 & 2376 & -1362 \\ -4248 & -444 & 798 & 3468 & -4962 & 546 & -1362 & 6204 \end{bmatrix}$$

$$A_3 = \frac{1}{245}\begin{bmatrix} 612 & 162 & 162 & -162 & 162 & -162 & -162 & -612 \\ 162 & 2082 & 342 & 252 & 342 & -24 & -1050 & -2106 \\ 162 & 342 & 2082 & -24 & 342 & -1050 & 252 & -2106 \\ -162 & 252 & -24 & 1554 & -1050 & -186 & -186 & -198 \\ 162 & 342 & 342 & -1050 & 2082 & 252 & -24 & -2106 \\ -162 & -24 & -1050 & -186 & 252 & 1554 & -186 & -198 \\ -162 & -1050 & 252 & -186 & -24 & -186 & 1554 & -198 \\ -612 & -2106 & -2106 & -198 & -2106 & -198 & -198 & 7524 \end{bmatrix}$$

Both of these matrices are positive semidefinite. We need that, for every tournament J on 5 vertices, the expression

$$\begin{aligned} &-8 \cdot t_{\text{inj}}(C_3, J) - \frac{1}{4} \cdot 1024 \cdot t_{\text{inj}}(H_{10}, J) \\ &\quad - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j} - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j} \end{aligned}$$

is equal to $-5/4$. The values of the coefficients can be found in the appendix at the end of the paper. We will only compute this expression for two examples of J for the sake of demonstration. First, in the case that $J = H_8$, it becomes

$$\begin{aligned} &-8 \cdot 0 - \frac{1}{4} \cdot 1024 \cdot 0 - \frac{2}{120}(A_2)_{1,1} - \frac{1}{120}(A_2)_{1,2} - \frac{1}{120}(A_2)_{1,4} \\ &- \frac{1}{120}(A_2)_{1,8} - \frac{1}{120}(A_2)_{2,1} - \frac{2}{120}(A_2)_{2,2} - \frac{1}{120}(A_2)_{2,4} - \frac{1}{120}(A_2)_{2,8} \\ &- \frac{1}{120}(A_2)_{4,1} - \frac{1}{120}(A_2)_{4,2} - \frac{2}{120}(A_2)_{4,4} - \frac{1}{120}(A_2)_{4,8} - \frac{1}{120}(A_2)_{8,1} \\ &\qquad - \frac{1}{120}(A_2)_{8,2} - \frac{1}{120}(A_2)_{8,4} - \frac{2}{120}(A_2)_{8,8} \end{aligned}$$

which, upon plugging the appropriate values from A_2, becomes

$$\begin{aligned} &-8 \cdot 0 - \frac{1}{4} \cdot 1024 \cdot 0 - \frac{2}{120} \cdot \frac{4724}{245} + \frac{1}{120} \cdot \frac{1883}{245} + \frac{1}{120} \cdot \frac{4598}{245} \\ &+ \frac{1}{120} \cdot \frac{4248}{245} + \frac{1}{120} \cdot \frac{1883}{245} - \frac{2}{120} \cdot \frac{6512}{245} - \frac{1}{120} \cdot \frac{3150}{245} + \frac{1}{120} \cdot \frac{444}{245} \\ &+ \frac{1}{120} \cdot \frac{4598}{245} - \frac{1}{120} \cdot \frac{3150}{245} - \frac{2}{120} \cdot \frac{5490}{245} - \frac{1}{120} \cdot \frac{3468}{245} + \frac{1}{120} \cdot \frac{4248}{245} \\ &\qquad + \frac{1}{120} \cdot \frac{444}{245} - \frac{1}{120} \cdot \frac{3468}{245} - \frac{2}{120} \cdot \frac{6204}{245} = -\frac{5}{4} \end{aligned}$$

As another example, consider $J = H_{12}$. The expression becomes

$$-8\cdot\frac{12}{60}-\frac{1}{4}\cdot 1024\cdot 0-\frac{3}{120}(A_2)_{3,5}-\frac{3}{120}(A_2)_{5,3}-\frac{3}{120}(A_2)_{6,7}-\frac{3}{120}(A_2)_{7,6}$$
$$-\frac{3}{120}(A_3)_{1,8}-\frac{3}{120}(A_3)_{2,6}-\frac{3}{120}(A_3)_{3,4}-\frac{3}{120}(A_3)_{4,3}-\frac{3}{120}(A_3)_{5,7}$$
$$-\frac{3}{120}(A_3)_{6,2}-\frac{3}{120}(A_3)_{7,5}-\frac{3}{120}(A_3)_{8,1}$$

which evaluates to

$$-8\cdot\frac{12}{60}-\frac{1}{4}\cdot 1024\cdot 0+\frac{3}{120}\cdot\frac{371}{245}+\frac{3}{120}\cdot\frac{371}{245}+\frac{3}{120}\cdot\frac{660}{245}+\frac{3}{120}\cdot\frac{660}{245}$$
$$+\frac{3}{120}\cdot\frac{612}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}$$
$$+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{612}{245}=-\frac{5}{4}.$$

Now, suppose that W is a tournamenton such that

$$8\cdot t(C_3,W)+\frac{1}{4}\cdot 1024\cdot t(H_{10},W)=\frac{5}{4}$$

and let us show that W is equal to $1/2$ almost everywhere. The kernel of A_2 is spanned by $(1,1,1,1,1,1,1,1)^T$. By the "moreover" part of Lemma 5.15, we must have that

$$t_3(F_1^2,W)(x_1,x_2,x_3)=t_3(F_2^2,W)(x_1,x_2,x_3)=\cdots=t_3(F_8^2,W)(x_1,x_2,x_3).$$

By Lemma 5.9, we have

$$\sum_{i=1}^{8}t(F_i^2,W)(x_1,x_2,x_3)=t(TT_3,W)(x_1,x_2,x_3)$$

for almost all $(x_1,x_2,x_3)\in[0,1]^3$. Putting these two facts together, we must have $t_3(F_1^2,W)(x_1,x_2,x_3)=t_3(F_7^2,W)(x_1,x_2,x_3)=\frac{1}{8}t(TT_3,W)(x_1,x_2,x_3)$ for almost all $(x_1,x_2,x_3)\in[0,1]^3$. Integrating out these two equalities over all $(x_1,x_2,x_3)\in[0,1]^3$ yields $t(TT_4,W)=\frac{1}{8}t(TT_3,W)$ and $t(C_4,W)=\frac{1}{8}t(TT_3,W)$, respectively. So, we get that $W=1/2$ almost everywhere by Lemma 5.11. □

Proof of Theorem 5.5. — We apply Lemma 5.15 with the following matrices:

$$A_2 = \frac{1}{7}\begin{bmatrix} 257 & -130 & -214 & 127 & -57 & 201 & 40 & -224 \\ -130 & 199 & 123 & -179 & 72 & -130 & -86 & 131 \\ -214 & 123 & 210 & -134 & 45 & -190 & -48 & 208 \\ 127 & -179 & -134 & 204 & -79 & 120 & 77 & -136 \\ -57 & 72 & 45 & -79 & 58 & -44 & -30 & 35 \\ 201 & -130 & -190 & 120 & -44 & 219 & 51 & -227 \\ 40 & -86 & -48 & 77 & -30 & 51 & 59 & -63 \\ -224 & 131 & 208 & -136 & 35 & -227 & -63 & 276 \end{bmatrix}$$

$$A_3 = \frac{1}{7}\begin{bmatrix} 184 & -31 & -31 & 22 & -31 & 22 & 22 & -157 \\ -31 & 62 & -10 & -26 & -10 & 4 & -11 & 22 \\ -31 & -10 & 62 & 4 & -10 & -11 & -26 & 22 \\ 22 & -26 & 4 & 61 & -11 & -11 & -11 & -28 \\ -31 & -10 & -10 & -11 & 62 & -26 & 4 & 22 \\ 22 & 4 & -11 & -11 & -26 & 61 & -11 & -28 \\ 22 & -11 & -26 & -11 & 4 & -11 & 61 & -28 \\ -157 & 22 & 22 & -28 & 22 & -28 & -28 & 175 \end{bmatrix}$$

These matrices are positive semidefinite. We need that, for every tournament J on 5 vertices, the expression

$$\begin{aligned} &-8 \cdot t_{\mathrm{inj}}(C_3, J) - \frac{1}{4} \cdot 1024 \cdot t_{\mathrm{inj}}(H_{11}, J) \\ &\quad - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j} - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j} \end{aligned}$$

is equal to $-5/4$. The values of the coefficients can be found in the appendix at the end of the paper. We will only compute this expression in one example for the sake of demonstration. In the case $J = H_{19}$, it becomes

$$\begin{aligned} &-8 \cdot \frac{15}{60} - \frac{1}{4} \cdot 1024 \cdot 0 - \frac{5}{120}(A_2)_{5,7} - \frac{5}{120}(A_2)_{7,5} - \frac{5}{120}(A_3)_{2,7} \\ &- \frac{5}{120}(A_3)_{3,6} - \frac{5}{120}(A_3)_{4,5} - \frac{5}{120}(A_3)_{5,4} - \frac{5}{120}(A_3)_{6,3} - \frac{5}{120}(A_3)_{7,2} \end{aligned}$$

which is

$$\begin{aligned} &-8 \cdot \frac{15}{60} - \frac{1}{4} \cdot 1024 \cdot 0 + \frac{5}{120} \cdot \frac{30}{7} + \frac{5}{120} \cdot \frac{30}{7} + \frac{5}{120} \cdot \frac{11}{7} \\ &\quad + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} = -\frac{5}{4}. \end{aligned}$$

Now, suppose that W is a regular tournamenton such that

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{11}, W) = \frac{5}{4}$$

and let us show that W is equal to $1/2$ almost everywhere. The kernel of A_2 is spanned by $(1,1,1,1,1,1,1,1)^T$. So, analogously to the proof of Theorem 5.4, we get that W is equal to $1/2$ almost everywhere. □

Proof of Theorem 5.6. — We apply Lemma 5.15 with the following matrix:

$$A_2 = \frac{1}{945} \begin{bmatrix} 3680 & 1296 & 1080 & -2376 & -6376 & 1400 & 2808 & -1512 \\ 1296 & 4528 & -1080 & -1512 & -4424 & -1400 & 5832 & -3240 \\ 1080 & -1080 & 5616 & -2160 & -3240 & 3240 & -5400 & 1944 \\ -2376 & -1512 & -2160 & 4320 & 5400 & -1512 & -3240 & 1080 \\ -6376 & -4424 & -3240 & 5400 & 16200 & -5400 & -5400 & 3240 \\ 1400 & -1400 & 3240 & -1512 & -5400 & 5400 & -3240 & 1512 \\ 2808 & 5832 & -5400 & -3240 & -5400 & -3240 & 16200 & -7560 \\ -1512 & -3240 & 1944 & 1080 & 3240 & 1512 & -7560 & 4536 \end{bmatrix}.$$

This matrix is positive semidefinite. We need that, for every tournament J on 5 vertices, the expression

$$8 \cdot t_{\mathrm{inj}}(TT_3, J) + \frac{1}{7} \cdot 1024 \cdot t_{\mathrm{inj}}(H_{13}, J) - \sum_{i=1}^{8} \sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j}$$

is equal to $8/7$. The values of the coefficients can be found in the appendix at the end of the paper. We will only compute this expression in one example for the sake of demonstration. In the case $J = H_{13}$, we get

$$\begin{aligned} 8 \cdot \frac{6}{60} + \frac{1}{7} \cdot 1024 \cdot \frac{1}{120} - \frac{1}{120}(A_2)_{1,7} - \frac{1}{120}(A_2)_{2,7} - \frac{1}{120}(A_2)_{4,5} \\ - \frac{1}{120}(A_2)_{5,4} - \frac{2}{120}(A_2)_{5,5} - \frac{1}{120}(A_2)_{5,8} - \frac{1}{120}(A_2)_{7,1} \\ - \frac{1}{120}(A_2)_{7,2} - \frac{2}{120}(A_2)_{7,7} - \frac{1}{120}(A_2)_{8,5} \end{aligned}$$

and plugging in yields

$$\begin{aligned} 8 \cdot \frac{6}{60} + \frac{1}{7} \cdot 1024 \cdot \frac{1}{120} - \frac{1}{120} \cdot \frac{2808}{945} - \frac{1}{120} \cdot \frac{5832}{945} - \frac{1}{120} \cdot \frac{5400}{945} \\ - \frac{1}{120} \cdot \frac{5400}{945} - \frac{2}{120} \cdot \frac{16200}{945} - \frac{1}{120} \cdot \frac{3240}{945} - \frac{1}{120} \cdot \frac{2808}{945} \\ - \frac{1}{120} \cdot \frac{5832}{945} - \frac{2}{120} \cdot \frac{16200}{945} - \frac{1}{120} \cdot \frac{3240}{945} = \frac{8}{7}. \end{aligned}$$

Now, suppose that W is a tournamenton such that

$$8 \cdot t(TT_3, W) + \frac{1}{7} \cdot 1024 \cdot t(H_{13}, W) = \frac{8}{7}$$

and let us show that W is equal to $1/2$ almost everywhere. The kernel of A_2 is spanned by $(1,1,1,1,1,1,1,1)^T$ and $(1,1,-1,-1,1,1,-1,-1)^T$. By Lemma 5.9, for almost every $(x_1,x_2,x_3) \in [0,1]^3$, we have

$$\sum_{i=1}^{8} t(F_i^2, W)(x_1,x_2,x_3) = t(TT_3, W)(x_1,x_2,x_3).$$

Combining these two facts with the "moreover" part of Lemma 5.15, we must have that, for almost every $(x_1,x_2,x_3) \in [0,1]^3$ and any $(i,j) \in \{1,2,4,5\} \times \{3,4,7,8\}$,

$$t(F_i^2, W)(x_1,x_2,x_3) + t(F_j^2, W)(x_1,x_2,x_3) = \frac{1}{4} t(TT_3, W)(x_1,x_2,x_3).$$

Applying this in the case $i = 5$ and $j = 7$ and integrating over all (x_1,x_2,x_3) yields

$$t(C_4, W) = \frac{1}{8} t(TT_3, W).$$

Also, setting $i = 1$ and $j = 4$ and integrating gives us

$$t(TT_4, W) = \frac{1}{8} t(TT_3, W).$$

So, we conclude that $W = 1/2$ almost everywhere by Lemma 5.11. □

Proof of Theorem 5.7. — We apply Lemma 5.15 with the following matrices:

$$A_1 = \frac{4}{5} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

and

$$A_3 = \frac{8}{15} \begin{bmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & 3 \\ -1 & 27 & -5 & -5 & -5 & -5 & -5 & -1 \\ -1 & -5 & 27 & -5 & -5 & -5 & -5 & -1 \\ -1 & -5 & -5 & 27 & -5 & -5 & -5 & -1 \\ -1 & -5 & -5 & -5 & 27 & -5 & -5 & -1 \\ -1 & -5 & -5 & -5 & -5 & 27 & -5 & -1 \\ -1 & -5 & -5 & -5 & -5 & -5 & 27 & -1 \\ 3 & -1 & -1 & -1 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

Both are positive semidefinite. We need that, for every tournament J on 5 vertices, the expression

$$8 \cdot t_{\mathrm{inj}}(TT_3, J) + \frac{1}{5} \cdot 1024 \cdot t_{\mathrm{inj}}(H_{14}, J) - \sum_{i=1}^{4}\sum_{j=1}^{4} b_2(F_i^1 \cdot F_j^1; J) \cdot (A_1)_{i,j} - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j}$$

is equal to $6/5$. Again, the values of the coefficients can be found in the appendix at the end of the paper. For $J = H_8$, the expression becomes

$$8 \cdot \frac{10}{60} + \frac{1}{15} \cdot 1024 \cdot 0 - \frac{10}{120} \cdot (A_1)_{1,1} - \frac{10}{120} \cdot (A_1)_{1,2} - \frac{10}{120} \cdot (A_1)_{1,4} - \frac{10}{120} \cdot (A_1)_{2,2} - \frac{10}{120} \cdot (A_1)_{2,4} - \frac{10}{120} \cdot (A_1)_{4,4} = \frac{4}{3} + 0 - \frac{1}{15} + \frac{1}{15} - \frac{1}{15} - \frac{1}{15} + \frac{1}{15} - \frac{1}{15} = \frac{6}{5}.$$

For $J = H_9$, it is

$$8 \cdot \frac{9}{60} + \frac{1}{15} \cdot 1024 \cdot 0 - \frac{6}{120} A_1(1,1) - \frac{18}{120} A_1(1,2) - \frac{6}{120} A_1(1,4) - \frac{6}{120} A_1(2,2) - \frac{6}{120} A_1(2,4) - \frac{12}{120} A_1(3,4) - \frac{6}{120} A_1(4,4) - \frac{6}{120} A_3(8,8) = \frac{6}{5} + 0 - \frac{1}{25} + \frac{3}{25} - \frac{1}{25} - \frac{1}{25} + \frac{1}{25} + \frac{2}{25} - \frac{1}{25} - \frac{2}{25} = \frac{6}{5}.$$

The rest of the calculations are similar.

Now, suppose that W is a tournamenton such that

$$8 \cdot t(TT_3, W) + \frac{1}{5} \cdot 1024 \cdot t(H_{14}, W) = \frac{6}{5}.$$

The kernel of A_1 is spanned by $(1,1,0,0)^T, (1,0,1,0)^T$ and $(0,1,0,1)^T$. By the "moreover" part of Lemma 5.15, this implies that

$$t_2(F_2^1, W)(x_1, x_2) + t_2(F_3^1, W)(x_1, x_2) = \frac{1}{2} t(H_1, W)(x_1, x_2) = \frac{1}{2} W(x_1, x_2)$$

for almost all $(x_1, x_2) \in [0,1]^2$. Integrating this out over all x_1 and x_2 yields

$$t(TT_3, W) + t(C_3, W) = \frac{1}{4}$$

which, when combined with (3.2), tells us that

$$t(C_3, W) = 1/8. \tag{5.6}$$

We have $\ker(A_3) = \operatorname{span}\{(1,1,1,1,1,1,1,1)^T, (1,0,0,0,0,0,0,-1)^T\}$. Applying Lemma 5.9 and the "moreover" part of Lemma 5.15 again, we have that for almost all $(x_1, x_2, x_3) \in [0,1]^3$,

$$t(F_2^3, W)(x_1, x_2, x_3) = \frac{1}{8} t(C_3, W)(x_1, x_2, x_3)$$

and

$$t(F_1^3, W)(x_1, x_2, x_3) + t(F_8^3, W)(x_1, x_2, x_3) = \frac{1}{4} t(C_3, W)(x_1, x_2, x_3).$$

Integrating over all x_1, x_2 and x_3 and applying (5.6) yields

$$t(C_4, W) = \frac{1}{8} t(C_3, W) = \frac{1}{64}$$

and

$$t(H_5, W) + t(H_7, W) = \frac{1}{4} t(C_3, W) = \frac{1}{32}.$$

Finally, using Lemma 5.8 and (5.1) and the fact that $\operatorname{aut}(TT_4) = \operatorname{aut}(C_4) = 1$ and $\operatorname{aut}(H_5) = \operatorname{aut}(H_7) = 3$, we get

$$\begin{aligned} &d(TT_4, W) + d(C_4, W) + d(H_5, W) + d(H_7, W) = 1 \\ &\qquad \implies 24t(TT_4, W) + 24t(C_4, W) + 8t(H_5, W) + 8t(H_7, W) = 1 \end{aligned}$$

which, since $t(C_4, W) = 1/64$ and $t(H_5, W) + t(H_7, W) = 1/32$, yields $t(TT_4, W) = 1/64$. We are now done by the fact that TT_4 forces quasirandomness (see Theorem 1.2). □

6. Conclusion

The Forcing Conjecture [11, 44] says that a graph H forces quasirandomness if and only if it is bipartite and contains a cycle. Unfortunately, the results of this paper do not seem to point to such a natural property which distinguishes the tournaments which force quasirandomness in regular tournaments in general. Nonetheless, we wonder whether a deeper investigation into such tournaments could uncover some sort of "pattern."

Problem 6.1. — *Classify tournaments H which force quasirandomness in regular tournaments.*

One way in which Problem 6.1 could end up being solved is if, as in the case of tournaments which force quasirandomness in general (not necessarily regular) tournaments, there end up being only finitely many non-transitive tournaments which force quasirandomness in regular tournaments. So, a first step toward Problem 6.1 is to determine whether or not an infinite family of such tournaments exists.

QUESTION 6.2. — *Are there infinitely many non-transitive tournaments which force quasirandomness in regular tournaments?*

More ambitiously, we ask the following.

QUESTION 6.3. — *Is it true that almost every tournament forces quasirandomness in regular tournaments?*

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Appendix A. Flag Algebra Coefficients

The purpose of this appendix is to list all of the coefficients $b_2(F_i^1, F_j^1; J)$ for $1 \leqslant i, j \leqslant 4$ and $b_3(F_i^q, F_j^q; J)$ for $1 \leqslant i, j \leqslant 8$ and $q \in \{2, 3\}$, where J is a tournament on 5 vertices and the flags F_i^q are as in Section 5. All of these coefficients were computed by computer but can be easily checked by hand. It will be convenient to record these coefficients in matrices. For every such J, let $B_2^1(J)$ be the 4×4 matrix in which the entry on the ith row and jth column is $b_2(F_i^1, F_j^1; J)$. Similarly, for every such J and $q \in \{2, 3\}$, let $B_3^q(J)$ be the 8×8 matrix in which the entry on the ith row and jth column is $b_3(F_i^q, F_j^q; J)$. The matrices are as follows.

For $J = H_8$, we have

$$B_2^1(H_8) = \frac{1}{120}\begin{bmatrix} 10 & 5 & 0 & 5 \\ 5 & 10 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 10 \end{bmatrix}$$

$$B_3^2(H_8) = \frac{1}{120}\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B_3^3(H_8) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_9$, we have

$$B_2^1(H_9) = \frac{1}{120}\begin{bmatrix} 6 & 9 & 0 & 3 \\ 9 & 6 & 0 & 3 \\ 0 & 0 & 0 & 6 \\ 3 & 3 & 6 & 6 \end{bmatrix}$$

$$B_3^2(H_9) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_9) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

For $J = H_{10}$, we have

$$B_2^1(H_{10}) = \frac{1}{120}\begin{bmatrix} 4 & 8 & 1 & 3 \\ 8 & 4 & 2 & 2 \\ 1 & 2 & 2 & 7 \\ 3 & 2 & 7 & 4 \end{bmatrix}$$

$$B_3^2(H_{10}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$B_3^3(H_{10}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For $J = H_{11}$, we have

$$B_2^1(H_{11}) = \frac{1}{120}\begin{bmatrix} 6 & 6 & 3 & 3 \\ 6 & 6 & 0 & 6 \\ 3 & 0 & 0 & 3 \\ 3 & 6 & 3 & 6 \end{bmatrix}$$

$$B_3^2(H_{11}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{11}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{12}$, we have

$$B_2^1(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 6 & 3 \\ 3 & 0 & 6 & 3 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 6 & 0 \end{bmatrix}$$

$$B_3^2(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{13}$, we have

$$B_2^1(H_{13}) = \frac{1}{120}\begin{bmatrix} 2 & 1 & 4 & 5 \\ 1 & 2 & 8 & 1 \\ 4 & 8 & 8 & 4 \\ 5 & 1 & 4 & 2 \end{bmatrix}$$

$$B_3^2(H_{13}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{13}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{14}$, we have

$$B_2^1(H_{14}) = \frac{1}{120}\begin{bmatrix} 4 & 2 & 3 & 5 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 6 & 3 \\ 5 & 2 & 3 & 4 \end{bmatrix}$$

$$B_3^2(H_{14}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_3^3(H_{14}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{15}$, we have

$$B_2^1(H_{15}) = \frac{1}{120}\begin{bmatrix} 4 & 2 & 7 & 3 \\ 2 & 4 & 2 & 8 \\ 7 & 2 & 2 & 1 \\ 3 & 8 & 1 & 4 \end{bmatrix}$$

$$B_3^2(H_{15}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{15}) = \frac{1}{120}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{16}$, we have

$$B_2^1(H_{16}) = \frac{1}{120}\begin{bmatrix} 6 & 3 & 6 & 3 \\ 3 & 6 & 0 & 9 \\ 6 & 0 & 0 & 0 \\ 3 & 9 & 0 & 6 \end{bmatrix}$$

$$B_3^2(H_{16}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{16}) = \frac{1}{120}\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{17}$, we have

$$B_2^1(H_{17}) = \frac{1}{120}\begin{bmatrix} 2 & 4 & 5 & 3 \\ 4 & 2 & 4 & 4 \\ 5 & 4 & 4 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$B_3^2(H_{17}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{17}) = \frac{1}{120}\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

For $J = H_{18}$, we have

$$B_2^1(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 6 & 3 \\ 3 & 0 & 6 & 3 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 6 & 0 \end{bmatrix}$$

$$B_3^2(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{19}$, we have

$$B_2^1(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 0 \\ 5 & 10 & 10 & 5 \\ 5 & 0 & 5 & 0 \end{bmatrix}$$

$$B_3^2(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

BIBLIOGRAPHY

[1] N. Alon & J. H. Spencer, *The probabilistic method*, fourth ed., Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, 2016.

[2] M. Bucić, E. Long, A. Shapira & B. Sudakov, Tournament quasirandomness from local counting, *Combinatorica* **41** (2021), no. 2, p. 175–208.

[3] D. Burke, B. Lidický, F. Pfender & M. Philips, Inducibility of 4-vertex tournaments, 2022, https://arxiv.org/abs/2103.07047v2.

[4] T. F. N. Chan, A. Grzesik, D. Král' & J. A. Noel, Cycles of length three and four in tournaments, *J. Comb. Theory, Ser. A* **175** (2020), article no. 105276 (23 pages), https://doi.org/10.1016/j.jcta.2020.105276.

[5] T. F. N. Chan, D. Král', J. A. Noel, Y. Pehova, M. Sharifzadeh & J. Volec, Characterization of quasirandom permutations by a pattern sum, *Random Struct. Algorithms* **57** (2020), no. 4, p. 920–939.

[6] F. Chung, From quasirandom graphs to graph limits and graphlets, *Adv. Appl. Math.* **56** (2014), p. 135–174.

[7] F. R. K. Chung & R. L. Graham, Quasi-random hypergraphs, *Random Struct. Algorithms* **1** (1990), no. 1, p. 105–124.

[8] ———, Quasi-random set systems, *J. Am. Math. Soc.* **4** (1991), no. 1, p. 151–196.

[9] ———, Quasi-random tournaments, *J. Graph Theory* **15** (1991), no. 2, p. 173–198.

[10] F. R. K. Chung, R. L. Graham & R. M. Wilson, Quasi-random graphs, *Combinatorica* **9** (1989), no. 4, p. 345–362.

[11] D. Conlon, J. Fox & B. Sudakov, An approximate version of Sidorenko's conjecture, *Geom. Funct. Anal.* **20** (2010), no. 6, p. 1354–1366.

[12] ———, Hereditary quasirandomness without regularity, *Math. Proc. Camb. Philos. Soc.* **164** (2018), no. 3, p. 385–399.

[13] D. Conlon, H. Hàn, Y. Person & M. Schacht, Weak quasi-randomness for uniform hypergraphs, *Random Struct. Algorithms* **40** (2012), no. 1, p. 1–38.

[14] J. N. Cooper, Quasirandom permutations, *J. Comb. Theory, Ser. A* **106** (2004), no. 1, p. 123–143.

[15] J. W. Cooper, D. Král', A. Lamaison & S. Mohr, Quasirandom Latin squares, *Random Struct. Algorithms* **61** (2022), no. 2, p. 298–308.

[16] L. N. Coregliano, R. F. Parente & C. M. Sato, On the maximum density of fixed strongly connected subtournaments, *Electron. J. Comb.* **26** (2019), no. 1, article no. 1.44 (48 pages), https://doi.org/10.37236/6557.

[17] L. N. Coregliano & A. A. Razborov, On the density of transitive tournaments, *J. Graph Theory* **85** (2017), no. 1, p. 12–21.

[18] ———, Natural quasirandomness properties, *Random Struct. Algorithms* **63** (2023), no. 3, p. 624–688.

[19] G. Crudele, P. Dukes & J. A. Noel, Six Permutation Patterns Force Quasirandomness, to appear in *Discrete Anal.*, 2023, https://arxiv.org/abs/2303.04776v3.

[20] D. Dellamonica, Jr. & V. Rödl, Hereditary quasirandom properties of hypergraphs, *Combinatorica* **31** (2011), no. 2, p. 165–182.

[21] W. T. Gowers, Quasirandomness, counting and regularity for 3-uniform hypergraphs, *Comb. Probab. Comput.* **15** (2006), no. 1-2, p. 143–184.

[22] ———, Quasirandom groups, *Comb. Probab. Comput.* **17** (2008), no. 3, p. 363–387.

[23] S. Griffiths, Quasi-random oriented graphs, *J. Graph Theory* **74** (2013), no. 2, p. 198–209.

[24] A. Grzesik, D. Il'kovič, B. Kielak & D. Král', Quasirandom-forcing orientations of cycles, *SIAM J. Discrete Math.* **37** (2023), no. 4, p. 2689–2716.

[25] A. Grzesik, D. Král', L. Lovász & J. Volec, Cycles of a given length in tournaments, *J. Comb. Theory, Ser. B* **158** (2023), p. 117–145.

[26] A. Grzesik, D. Král' & O. Pikhurko, Forcing generalised quasirandom graphs efficiently, *Comb. Probab. Comput.* **33** (2024), no. 1, p. 16–31.

[27] R. Hancock, A. Kabela, D. Král', T. Martins, R. Parente, F. Skerman & J. Volec, No additional tournaments are quasirandom-forcing, *Eur. J. Comb.* **108** (2023), article no. 103632 (10 pages).

[28] S. Kalyanasundaram & A. Shapira, A note on even cycles and quasirandom tournaments, *J. Graph Theory* **73** (2013), no. 3, p. 260–266.

[29] M. G. Kendall & B. Babington Smith, On the method of paired comparisons, *Biometrika* **31** (1940), p. 324–345.

[30] Y. Kohayakawa, V. Rödl & J. Skokan, Hypergraphs, quasi-randomness, and conditions for regularity, *J. Comb. Theory, Ser. A* **97** (2002), no. 2, p. 307–352.

[31] D. Král', J.-B. Lee & J. A. Noel, Forcing quasirandomness with 4-point permutations, 2024, https://arxiv.org/abs/2407.06869v1.

[32] D. Král' & O. Pikhurko, Quasirandom permutations are characterized by 4-point densities, *Geom. Funct. Anal.* **23** (2013), no. 2, p. 570–579.

[33] M. Kurečka, Lower bound on the size of a quasirandom forcing set of permutations, *Comb. Probab. Comput.* **31** (2022), no. 2, p. 304–319.

[34] N. Linial & A. Morgenstern, On the number of 4-cycles in a tournament, *J. Graph Theory* **83** (2016), no. 3, p. 266–276.

[35] L. Lovász, *Combinatorial problems and exercises*, North-Holland, 1979, 551 pages.

[36] ———, *Large networks and graph limits*, Colloquium Publications, vol. 60, American Mathematical Society, 2012.

[37] L. Lovász & V. T. Sós, Generalized quasirandom graphs, *J. Comb. Theory, Ser. B* **98** (2008), no. 1, p. 146–163.

[38] L. Lovász & B. Szegedy, Limits of dense graph sequences, *J. Comb. Theory, Ser. B* **96** (2006), no. 6, p. 933–957.

[39] J. Ma & T. Tang, Minimizing cycles in tournaments and normalized q-norms, *Comb. Theory* **2** (2022), no. 3, article no. 6 (19 pages), https://doi.org/10.5070/C62359154.

[40] E. Moss & J. A. Noel, Off-Diagonal Ramsey Multiplicity, 2023, https://arxiv.org/abs/2306.17388v1.

[41] A. A. Razborov, Flag algebras, *J. Symb. Log.* **72** (2007), no. 4, p. 1239–1282.

[42] V. Rödl, On universality of graphs with uniformly distributed edges, *Discrete Math.* **59** (1986), no. 1-2, p. 125–134.

[43] A. Sah & M. Sawhney, The intransitive dice kernel: $\frac{1_{x\geqslant y}-1_{x\leqslant y}}{4} - \frac{3(x-y)(1+xy)}{8}$, 2023, https://arxiv.org/abs/2302.11293v1.

[44] J. Skokan & L. Thoma, Bipartite subgraphs and quasi-randomness, *Graphs Comb.* **20** (2004), no. 2, p. 255–262.

[45] A. Thomason, Pseudorandom graphs, in *Random graphs '85 (Poznań, 1985)*, North-Holland Mathematics Studies, vol. 144, North-Holland, 1987, p. 307–331.

[46] ———, Random graphs, strongly regular graphs and pseudorandom graphs, in *Surveys in combinatorics 1987 (New Cross, 1987)*, London Mathematical Society Lecture Note Series, vol. 123, Cambridge University Press, 1987, p. 173–195.

[47] E. Thörnblad, Decomposition of tournament limits, *Eur. J. Comb.* **67** (2018), p. 96–125.

[48] Y. Zhao & Y. Zhou, Impartial digraphs, *Combinatorica* **40** (2020), no. 6, p. 875–896.

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