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A STRUCTURAL DUALITY FOR PATH-DECOMPOSITIONS INTO PARTS OF SMALL RADIUS

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Abstract. — It is an easy observation that if a graph G admits a path-decomposition whose parts have small radius, then G contains no large subdivision of $K_{1,3}$ or K^3 as a (quasi-)geodesic subgraph. We show that these are in fact the only obstructions to such path-decompositions of small radial width, and we prove analogous results for decompositions modelled on cycles and subdivided stars instead of paths.

With our results we confirm in a strong form a conjecture of Georgakopoulos and Papasoglu on fat-minor-characterisations of graphs quasi-isometric to paths, cycles and paths, and subdivided stars, respectively. For this, we present a novel view on quasi-isometries between graphs by graph-decompositions of bounded radial width and spread. This new perspective enables us to prove further results in coarse graph theory, and may thus be of independent interest.

1. Introduction

1.1. Fat minors and quasi-isometries

Following Gromov's ideas on coarse geometry [19] into the realm of graphs, Georgakopoulos and Papasoglu [18] suggested a study of graphs from a coarse or metric perspective, which revolves around the concept of *quasi-isometry*. Roughly speaking, two metric spaces are quasi-isometric if

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their large-scale geometry coincides, and more formally, a quasi-isometry is a generalisation of bi-Lipschitz maps that allows for an additive error. For example, all locally finite Cayley graphs of a finitely generated group are quasi-isometric to each other (see e.g. [22, Proposition 5.2.5]).

As their favorite problem of metric graph-theoretic flavour, Georgakopoulos and Papasoglu proposed the following conjecture, whose qualitative converse is immediate:

Conjecture 1.1 ([18, Conjecture 1.1]). — *Let $\mathcal{X}$ be a finite set of finite graphs. Then there exists a function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that, for all integers K, every graph with no K-fat X minor for any $X \in \mathcal{X}$ is $f(K)$-quasi-isometric to a graph with no X minor for any $X \in \mathcal{X}$.*

Here, "K-fat minors" are a metric variant of minors: Roughly speaking, a K-fat X minor is an X minor with additional distance constraints: its branch sets and branch paths are pairwise at least K apart, except for incident vertex-edge pairs (see Section 1.4 for the formal definition).

Georgakopoulos and Papasoglu [18] verified their conjecture for $\mathcal{X} = \{K^3\}$ (which describes graphs that are quasi-isometric to a forest). An earlier result of Chepoi, Dragan, Newman, Rabinovich and Vaxes [9] yields the case $\mathcal{X} = \{K_{2,3}\}$ (quasi-isometric to an outerplanar graph). Fujiwara and Papasoglu [17] solved the $\mathcal{X} = \{K_4^-\}$-case (quasi-isometric to a cactus); see also [5] for a simpler proof.

Between the first appearance of this paper as a preprint on arXiv [2] and the publication of this journal version, some more results on Conjecture 1.1 have been obtained: Georgakopoulos and Papasoglu [18] verified Conjecture 1.1 for $\mathcal{X} = \{K_{1,m}\}$. Moreover, Albrechtsen, Jacobs, Knappe and Wollan [5] solved the case $\mathcal{X} = \{K^4\}$, and Albrechtsen, Distel and Georgakopoulos [3] proved the case $\mathcal{X} = \{K_{2,t}\}$ for all $t \in \mathbb{N}$. In contrast to these positive results, Davies, Hickingbotham, Illingworth and McCarty [10] showed that Conjecture 1.1 is false in general. In fact, Albrechtsen, Distel and Georgakopoulos [4] showed that Conjecture 1.1 already fails for several small graphs such as $\mathcal{X} = \{K_{2,2,2}\}$.

In this paper, we establish Conjecture 1.1 for three further small cases but in a stronger form:

Theorem 1. — *Conjecture 1.1 is true for $\mathcal{X} = \{K^3, K_{1,3}\}$ (quasi-isometric to a disjoint union of paths), $\mathcal{X} = \{K_{1,3}\}$ (cycles and paths)*[(1)]

(1) Georgakopoulos and Papasoglu prove the more general case $\mathcal{X} = \{K_{1,m}\}$ in a second version of their paper [18], which appeared after the first appearance of this paper as a preprint on arXiv [2].

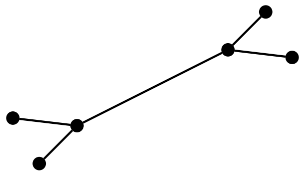

Figure 1.1. The wrench *graph W.*

and $\mathcal{X} = \{K^3, W\}$ (subdivided stars), where W is the graph depicted in Figure 1.1. In all these cases, Conjecture 1.1 even holds true when replacing "fat minors" by "quasi-geodesic topological minors".

We refer the reader to Theorems 2–4 for the respective functions f.

In Theorem 1, "quasi-geodesic topological minors" are a metric version of topological minors, whose model in G is "quasi-geodesically" embedded (see Section 1.4 for the definition). Roughly speaking, the metric of the topological minor agrees (up to a multiplicative constant) with the metric induced by the graph G. In particular, quasi-geodesic topological minors yield fat minors but in general not the other way around (Lemma 3.14).

Our work on Theorem 1 was independent of Conjecture 1.1 and the involved concepts such as fat minors and quasi-isometries. In fact, we only discovered the connections of our results to quasi-isometries and fat minors, and in particular to Conjecture 1.1, through [18]. Our approach is not based on quasi-isometries, but on the notion of graph-decompositions, which we describe in what follows.

1.2. Graph-decompositions

Graph-decompositions [13] are a natural extension of tree-decompositions which allow the bags V_h of decompositions $(H, \mathcal{V})$ to be arranged along general decomposition graphs H instead of just trees. We then say that $(H, \mathcal{V})$ is *modelled on H* and call it an *H-decomposition*. Moreover, given any graph class $\mathcal{H}$, we refer to H-decompositions with $H \in \mathcal{H}$ as *$\mathcal{H}$-decompositions*. Recent applications of graph-decompositions include a local-global decomposition theorem [13] as well as the study of local separations [20] and of locally chordal graphs [1].

In this paper, we present a width-notion for graph-decompositions such that a graph G has an H-decomposition of small width if and only if H resembles the large-scale structure of G. The naive approach defines the "width" of a graph-decomposition analogously to tree-width, that is, as the minimal cardinality of a bag of the decomposition (minus 1). However, the respective "$\mathcal{H}$-width", the minimal width of an $\mathcal{H}$-decomposition for a given graph class $\mathcal{H}$, does not yield a meaningful extension of tree-width: if a minor-closed class $\mathcal{H}$ of graphs has bounded tree-width, then every graph of small $\mathcal{H}$-width has small tree-width itself, and if $\mathcal{H}$ has unbounded tree-width, then the $\mathcal{H}$-width of every graph is at most 2 (minus 1) [14, 25].(2)

This inspired us to consider a metric perspective instead: To define the "width" of a graph-decomposition, we measure the size of its bags not in terms of their cardinality, but by the radius of its *parts*, the induced subgraphs on the bags of the decomposition. More formally, recall that the *radius* of a graph G is the smallest $r \in \mathbb{N}$ such that some vertex of G has distance at most r to all vertices of G. We then let the *(inner-)radial width* of a decomposition be the largest radius among its parts and define the *radial $\mathcal{H}$-width* of G for a given class $\mathcal{H}$ of graphs to be the smallest radial width among all decompositions of G modelled on graphs in $\mathcal{H}$.

This notion of radial width is not new for tree-decompositions. Indeed, the radial $\mathcal{H}$-width for the class $\mathcal{H}$ of all trees, or *radial tree-width* for short, has been studied before, e.g. as the equivalent *tree-length* in [15] and *tree-breadth* in [16]. Similar to the classical tree-width notion [24], several computationally hard problems such as the computation of the metric dimension of a graph [6] can be efficiently solved on graphs of small radial tree-width (for a summary, see [15, Section 1] or [21]).

1.3. Interplay between graph-decompositions and quasi-isometries

As it turns out, graph-decompositions and quasi-isometries are closely related. In fact, these notions become qualitatively equivalent if we restrict to "honest" graph-decompositions of bounded radial width and bounded "radial spread".

An H-decomposition of G is *honest* if all its bags are non-empty and for every edge of its decomposition graph H, the bags corresponding to its endnodes intersect. Informally speaking, being honest ensures that all connectivity in the decomposition graph H also appears in the decomposed graph G. For each vertex $v \in G$ let H_v be the induced subgraph of H on the

(2) By Diestel and Kühn [14, Proposition 3.7], every graph G has a grid-decomposition of width at most 2 (minus 1). Thus, the $\mathcal{H}$-width of G is at most 2 (minus 1) for any graph class $\mathcal{H}$ of unbounded tree-width by Robertson and Seymour's Grid Theorem [25].

set of all nodes $h \in H$ whose corresponding bags contain v. The *(inner-) radial spread* of the H-decomposition is then defined as the largest radius of the H_v with $v \in V(G)$.

The equivalence between the existence of an honest H-decomposition with bounded radial width and spread and of a bounded quasi-isometry to H was observed for the case of trees H by Berger and Seymour [8, 4.1]. Here, we extend their observation to arbitrary graphs H.

Proposition 1.2. — *There exist functions $g : \mathbb{N}^2 \to \mathbb{N}^2$ and $h_1, h_2 : \mathbb{N}^2 \to \mathbb{N}$ such that the following holds for all graphs G, H:*

(1) If G admits an honest H-decomposition of radial width r_0 and radial spread r_1, then G is $g(r_0, r_1)$-quasi-isometric to H.
(2) If G is (L, C)-quasi-isometric to H, then G admits an honest H-decomposition of radial width $h_1(L, C)$ and radial spread $h_2(L, C)$.

We refer the reader to Section 3.3 for the detailed statement on the functions g and h_1, h_2.

The correspondence given by Proposition 1.2 paves the way for a new proof method towards Conjecture 1.1: one through the graph-theoretic construction of a suitable graph-decomposition. We follow this approach in the present paper and further demonstrate its power and versatility in [5].

Proposition 1.2 in particular implies that we reached our initial goal from the beginning of Section 1.2 with the notions of “radial width” and “radial spread”: a graph H resembles the large-scale geometry of G (in terms of bounded quasi-isometries) if and only if G admits an honest H-decomposition of bounded radial width and spread.

1.4. Our results

With a suitable width measure at hand, our next goal was to identify obstructions to small radial $\mathcal{H}$-width and to characterise which graphs have small radial $\mathcal{H}$-width for given classes $\mathcal{H}$ of graphs. Similar to Conjecture 1.1, we considered metric versions of minors as candidates for suitable obstructions. For the graph classes $\mathcal{H}$ that we study in this paper, our obstructions are “quasi-geodesic topological minors”, which can be seen as a special case of fat minors. Let us make this precise in what follows.

A $(\geqslant k)$-*subdivision* of a graph X is a graph X' which arises from X by subdividing each edge at least k times, i.e. replacing every edge with a path of at least $k+1$ edges. Further, a subgraph X of a graph G is *c-quasi-geodesic*[(3)] for some $c \in \mathbb{N}$ if the distance of any two vertices x and y in X is at most c times their distance in G. Here, the parameter c describes how well the distances in X approximate the distances in G; in particular, a subgraph is geodesic if and only if it is 1-quasi-geodesic.

Quasi-geodesic topological minors are indeed an obstruction to small radial width: if a $(\geqslant k)$-subdivision of a graph X is a c-quasi-geodesic subgraph of G, then X is a $\lfloor \frac{k-2}{2c} \rfloor$-fat minor of G (Lemma 3.14). Hence, as fat minors form an obstruction to small radial width (Lemma 3.15), so do quasi-geodesic topological minors (Proposition 3.16).

Our main result, Theorem 1, asserts that these are in fact the only obstructions to small radial $\mathcal{H}$-width for the three cases where $\mathcal{H}$ consists of paths, cycles and paths, and subdivided stars, respectively. We give the detailed statements in the following Theorems 2–4, from which we then deduce Theorem 1.

Theorem 2 (Radial path-width). — *Let $k \in \mathbb{N}$. If a connected graph G contains no $(\geqslant k)$-subdivision of K^3 as a geodesic subgraph and no $(\geqslant 3k)$-subdivision of $K_{1,3}$ as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a path P of radial width at most $18k+2$ and radial spread at most $18k+1$.*

Moreover, P is $(1, 18k+2)$-quasi-isometric to G.

Theorem 3 (Radial cycle-width). — *Let $k \in \mathbb{N}$. If a connected graph G contains no $(\geqslant 3k)$-subdivision of $K_{1,3}$ as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a cycle or path C of radial width at most $18k+2$ and radial spread at most $36k+2$.*

Moreover, C is $(1, 18k+2)$-quasi-isometric to G.

Theorem 4 (Radial star-width). — *Let $k \in \mathbb{N}$. If a connected graph G contains no $(\geqslant k)$-subdivision of K^3 as a geodesic subgraph and no $(\geqslant 3k)$-subdivision of the wrench graph W as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a subdivided star S of radial width at most $58k+9$ and radial spread at most $30k+7$.*

Moreover, there exists some $C_k \in \mathbb{N}$ such that some subdivided star is $(1, C_k)$-quasi-isometric to G.[(4)]

(3) Note that in metric spaces this property is often called $(c, 0)$-quasi-geodesic. In [7] it is called c-multiplicative.

(4) While we will only formally prove that some subdivided star S' and some $C_k \in \mathbb{N}$ exists, one can check by carefully reading the proof that we may choose $S' = S$ and $C_k = 60k+14$ (see the paragraph after the proof of Theorem 4 in Section 6 for details).

Proof of Theorem 1 given Theorems 2–4. — Let $K \in \mathbb{N}$, and let G be a graph with no K-fat X minor for any X in the respective $\mathcal{X}$. By Lemma 3.14 there is an integer k depending on K and $\mathcal{X}$ only, such that, for every $c \in \mathbb{N}$, G contains no $(\geqslant c \cdot k)$-subdivision of any $X \in \mathcal{X}$ as c-quasi-geodesic subgraph. Now we deduce Theorem 1 from Theorems 2–4 by applying the respective theorem to the components of G and combine the obtained $(1, C_k)$-quasi-isometries to one from the disjoint union H of their domains to G. Lemma 3.8 yields the desired $f(K)$-quasi-isometry from G to H. □

1.5. Sketch of the proofs

Let us give a brief overview of the proof techniques for Theorems 2–4. For the proof of Theorem 2 (radial path-width), we start with a longest geodesic path P in the connected graph G. We then show that either balls of small radius around $V(P)$ cover all of G or we can find a geodesic $(\geqslant k)$-subdivision of the triangle K^3 or a 3-quasi-geodesic $(\geqslant 3k)$-subdivision of the claw $K_{1,3}$. In the former case we construct a P-decomposition of G by letting the bag corresponding to a node $p \in P$ be the union of all those small radius-balls, whose centre vertex has small distance to p in P.

The proof technique for Theorem 2 immediately generalises to Theorem 3 (radial cycle-width). The proof of Theorem 4 (radial star-width) is more involved. As before we start with a longest geodesic path P in G and consider the subgraph G' of G which is covered by balls of small radius around $V(P)$. Unlike in the path-case, G' will in general not be equal to G. But if G does not contain a large subdivision of K^3 or W as a (3-quasi-)geodesic subgraph, then every component of $G - G'$ will have its neighbours in G' only close to the same vertex p_s of P. We then identify a geodesic path within each component of $G - G'$ such that all vertices in the component have small distance to this path. All these paths can then be combined with P into a subdivided star S, by adding edges between their last vertices and p. We then assign to each node s of S a suitable bag V_s of vertices of bounded distance to s.

1.6. An open conjecture about quasi-isometries to trees

For the special case of $\mathcal{X} = \{K^3\}$, Georgakopoulos and Papasoglu [18] answered Conjecture 1.1 in the affirmative, proving that the absence of a K-fat K^3 minor implies the existence of an $f(K)$-quasi-isometry to a forest. Berger and Seymour [8] characterised the graphs quasi-isometric to a forest using a similar kind of obstruction. Our results Theorems 2–4 yield another natural guess for such an obstruction – long quasi-geodesic cycles:

Conjecture 1.3. — *There is a function* $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ *such that if a connected graph* G *does not contain a* c*-quasi-geodesic cycle of length at least* $3ck$ *for some* $c, k \in \mathbb{N}$*, then* G *is* $f(k)$*-quasi-isometric to a tree.*

If true, this statement would strengthen the respective results of Georgakopoulos and Papasoglu and of Berger and Seymour. Note that by Proposition 1.2 and [8, 4.1] we can equivalently ask about the existence of a function f which guarantees that G admits an honest tree-decomposition of radial width at most $f(k)$.

1.7. Structure of the paper

Section 2 collects some basic definitions. In Section 3 we introduce the notion of graph-decompositions and their radial width and spread, and show how they are related to quasi-isometries, proving Proposition 1.2. We also prove in Section 3.4 that quasi-geodesic topological minors yield fat minors and are an obstruction to small radial width. The following three Sections 4–6 contain the proofs of Theorems 2–4, respectively. In the appendix of the arXiv version we discuss further problems resulting from our work. In particular, we show that the proof technique described in Section 1.5 for Theorems 2 and 3 does not work for Conjecture 1.3, and we show that the strengthening of Conjecture 1.1 for quasi-geodesic topological minors fails earlier than Conjecture 1.1 itself.

2. Preliminaries

We mainly follow the notations from [11]. We recall that for two sets X, Y of vertices of a graph G an X–Y *path* in G is a path in G whose only vertex in X is its first vertex and whose only vertex in Y is its last vertex. Moreover, for a subgraph H of G an H*-path* in G is a non-trivial path in G which meets H precisely in its endvertices. For the remainder of the section, we briefly present the definitions we will require going forward. All graphs in this paper are finite.

Let G be a graph. We write $d_G(v,u)$ for the distance of the two vertices v and u in G. For two sets U and U' of vertices of G, we write $d_G(U,U')$ for the minimum distance of two elements of U and U', respectively. If one of U or U' is just a singleton, then we may omit the braces. Further, if X is a subgraph of G, then we abbreviate $d_G(U,V(X))$ as $d_G(U,X)$ for notational simplicity.

Given a set X of vertices of G, the *r-ball (in G) around X*, denoted as $B_G(X,r)$, is the set of all vertices in G of distance at most r from X in G. If $X=\{v\}$ for some $v \in V(G)$, then we omit the braces, writing $B_G(v,r)$ for the r-ball (in G) around v.

Further, the *radius* $\mathrm{rad}(G)$ of G is the smallest number $k \in \mathbb{N}$ such that there exists some vertex $v \in G$ with $d_G(v,w) \leqslant k$ for every vertex $w \in G$. Note that G has radius at most k if and only if there is some vertex $v \in G$ with $V(G) = B_G(v,k)$. If G is not connected, then we define its radius to be ∞. Additionally, if $U \subseteq V(G)$, then the *radius of U in G* is the smallest number $k \in \mathbb{N}$ such that there exists some vertex $v \in G$ with $U \subseteq B_G(v,k)$.

3. Graph-decompositions and quasi-isometries

In this section we first introduce the concept of graph-decompositions and their radial width and spread and collect some properties of these notions. We then study the connection between graph-decompositions of small radial width and spread and quasi-isometries to their decomposition graph, proving Proposition 1.2. Finally, we define quasi-geodesic topological minors and show that they yield fat minors, which indeed form obstruction to small radial width.

3.1. Graph-decompositions

Let us recall the notion of graph-decompositions:

Definition 3.1 (Graph-decomposition [13]). — *Let G and H be graphs and let $\mathcal{V} = (V_h)_{h \in H}$ be a family of sets V_h of vertices of G.*[5] *We call $(H,\mathcal{V})$*

[5] In [13] a graph-decomposition is, more generally, defined as a pair $(H,(G_h)_{h\in H})$ of a graph H and a family of (not necessarily induced) subgraphs G_h of G indexed by the nodes of H, instead of a family of vertex sets V_h as in this paper.

*an H-*decomposition *of G, a* decomposition of G modelled on H, *or just a* graph-decomposition, *if*

(H1) $\bigcup_{h\in H} G[V_h] = G$, *and*

(H2) for every vertex $v \in G$, the graph of $H_v := H[\{h \in H \mid v \in V_h\}]$ is connected.

The sets V_h are called the bags *of this graph-decomposition, their induced subgraphs $G[V_h]$ are its* parts, *and the graph H is its* decomposition graph. *Whenever a graph-decomposition is introduced as $(H, \mathcal{V})$, we tacitly assume $\mathcal{V} = (V_h)_{h\in H}$. For a class $\mathcal{H}$ of graphs, we call $(H, \mathcal{V})$ an* $\mathcal{H}$-decomposition *if it is an H-decomposition with $H \in \mathcal{H}$.*

A graph-decomposition $(H, (V_h)_{h\in H})$ of a graph G is *honest* if all its bags V_h are non-empty and $V_h \cap V_{h'} \neq \emptyset$ for every edge $hh' \in H$.

If G is connected and every bag of $(H, \mathcal{V})$ is non-empty, then it follows from Definition 3.1 that H has to be connected as well. More generally, we have the following lemma:

Lemma 3.2. — *Let $(H, \mathcal{V})$ be a graph-decomposition of a graph G. Then for any connected subgraph X of G, the graph $H_X := H[\{h \in V(H) \mid X \cap V_h \neq \emptyset\}]$ is connected.*

Proof. — (H1) yields for every edge vw in H a part $G[V_h]$ that contains vw. This implies that both H_v and H_w contain h and hence intersect. Since H_v and H_w are connected by (H2), it follows that $H_v \cup H_w$ is connected as well. This implies that as X is connected, $H_X = \bigcup_{x\in X} H_x$ is connected as well. □

The next lemma shows that we can obtain a new graph-decomposition by enlarging all bags of a given graph-decomposition simultaneously in that we replace each of its bags by the r-ball around it for some globally fixed $r \in \mathbb{N}$. A version of Lemma 3.3 for tree-decompositions was first proven in [12, Lemma A.1].

Lemma 3.3. — *Let $(H, \mathcal{V})$ be a graph-decomposition of a graph G, and let $r \in \mathbb{N}$. For every vertex $h \in H$, we let $V'_h := B_G(V_h, r)$. Then $(H, \mathcal{V}')$ is again a graph-decomposition of G.*

Proof. — For notational simplicity, we adopt the following conventions for the course of this proof. Given a subgraph X of G, we write $H_X := H[\{h \in H \mid V_h \cap V(X) \neq \emptyset\}]$ and $H'_X := H[\{h \in H \mid V'_h \cap V(X) \neq \emptyset\}]$.

By definition, $(H, \mathcal{V}')$ satisfies (H1). For (H2) consider any vertex $v \in G$. Then H_v is connected by (H2) and non-empty by (H1). So in order to prove that H'_v is connected as well, it suffices to show that for every $h \in V(H'_v)$, there is an h–$V(H_v)$ path in H'_v.

By the definition of V'_h, there exists $w \in V_h$ with $d_G(v, w) \leqslant r$, and we fix a shortest v–w path P in G. Then every vertex $p \in P$ satisfies $d_G(v, p) \leqslant r$ as witnessed by P. In particular, every node $h' \in H$ with $V_{h'} \cap V(P) \neq \emptyset$ satisfies $v \in V'_{h'} = B_G(V_{h'}, r)$, and hence H_P is a subgraph of H'_v. But now $h \in V(H_P)$ as P meets the vertex $w \in V_h$, and H_v is a subgraph of H_P as $v \in V(P)$. So since H_P is connected by Lemma 3.2, there exists a h–$V(H_v)$ path in H_P and hence in H'_v, as desired. □

3.2. Radial width and radial spread

While the usual width of a tree-decomposition is measured in terms of the cardinality of its bags, the radial width of a graph-decomposition is measured in terms of the radius of its parts as follows.

Definition 3.4 (Radial width). — *Let $(H, \mathcal{V})$ be a graph-decomposition of a graph G modelled on a graph H. The* (inner-)radial width *of $(H, \mathcal{V})$ is*

$$\operatorname{radw}(H, \mathcal{V}) := \max_{h \in V(H)} \operatorname{rad}(G[V_h]).$$

Note that if $G[V_h]$ is disconnected for some $h \in H$, then $\operatorname{radw}(H, \mathcal{V}) = \infty$. The outer-radial width *of $(H, \mathcal{V})$ is $\max_{h \in V(H)} \operatorname{rad}_G(V_h)$. We remark that the outer-radial width is at most the (inner-)radial width.*

Given a non-empty class $\mathcal{H}$ of graphs, the (inner-)radial $\mathcal{H}$-width *of G is*

$$\operatorname{radw}_{\mathcal{H}}(G) := \min \left\{ \operatorname{radw}(H, \mathcal{V}) \;\middle|\; \begin{array}{l} (H, \mathcal{V}) \text{ is a graph-decomposition of } G \\ \text{with } H \in \mathcal{H} \end{array} \right\}.$$

Note that the (inner-)radial $\mathcal{H}$-width will always be at most $\operatorname{rad}(G)$ for a connected graph G if $\mathcal{H}$ contains at least one non-empty graph. The outer-radial $\mathcal{H}$-width is defined analogously.

For classes $\mathcal{H}$ of graphs that we frequently use in this paper, we name the radial $\mathcal{H}$-width as in Table 3.1.

Under the name "tree breadth", the concept "outer-radial tree-width" has previously been studied [16, 21] with applications to tree-spanners and routing problems.

Definition 3.5 (Radial spread). — *Let $(H, \mathcal{V})$ be a graph-decomposition of a graph G modelled on a graph H. The* (inner-)radial spread *of $(H, \mathcal{V})$ is*

$$\operatorname{rads}(H, \mathcal{V}) := \max_{v \in V(G)} \operatorname{rad}(H_v).$$

Table 3.1. Nomenclature for frequently used classes $\mathcal{H}$ of graphs. The second column gives the list of graphs which defines the minor-closure of $\mathcal{H}$ via forbidden (topological) minors.

$\mathcal{H}$ comprises unions of	**Forbidden minors**	**$\mathcal{H}$-decomposition**	**radial $\mathcal{H}$-width**
paths	K^3, $K_{1,3}$	path-decomposition	radial path-width
subdivided stars	K^3, W	star-decomposition	radial star-width
trees	K^3	tree-decomposition	radial tree-width
cycles	$K_{1,3}$	cycle-decomposition	radial cycle-width

The outer-radial spread *of* $(H, \mathcal{V})$ *is* $\max_{v \in V(G)} \operatorname{rad}_H(H_v)$. *Note that the outer-radial spread is at most the (inner-)radial spread.*

We remark that the “outer” versions of radial width and spread are only used in Lemmata 3.6, 3.9 and 3.10. Hence, we will usually omit the “inner” when talking about (inner)-radial width and spread.

Berger and Seymour [8, 1.5] proved that the (inner-)radial tree-width of a graph is at most two times its outer-radial width, and thus these width measures are qualitatively equivalent. The next lemma generalises this to arbitrary decomposition graphs.

Lemma 3.6. — *Let $(H, \mathcal{V})$ be a graph-decomposition of a graph G of outer-radial width $k \in \mathbb{N}$ and (inner-) radial spread $r \in \mathbb{N}$. Then setting $V'_h := B_G(V_h, k)$ for every node $h \in H$ yields a graph-decomposition $(H, \mathcal{V}')$ of G of (inner-)radial width at most $2k$ and (inner-)radial spread at most $2kr$.*

Proof. — By Lemma 3.3, $(H, \mathcal{V}')$ is indeed a graph-decomposition of H. To show that $(H, \mathcal{V}')$ has (inner-)radial width at most $2k$, consider any node $h \in H$. Since $(H, \mathcal{V})$ has outer-radial width at most k, there is a vertex z_h with $V_h \subseteq B_G(z_h, k)$. Then $z_h \in V'_h$ and $G[V'_h]$ contains a shortest path in G between every vertex $x \in V_h$ and z_h. Moreover, there exists for any vertex $x' \in V'_h$ a vertex $x \in V_h$ with $d_{G[V'_h]}(x', x) = d_G(x', x) \leqslant k$. Thus, we have $d_{G[V'_h]}(z_h, x) \leqslant d_{G[V'_h]}(z_h, x') + d_{G[V'_h]}(x', x) \leqslant k + k = 2k$. This implies that each $G[V'_h]$ has radius at most $2k$, and hence $(H, \mathcal{V}')$ has (inner-)radial width at most $2k$.

To see that $(H, \mathcal{V}')$ has (inner-)radial spread at most $2kr$, consider any vertex v of G and any node h of H such that $v \in V'_h$. By the definition of V'_h, there exists a vertex $u \in V_h$ such that $d_G(v, u) \leqslant k$. Let $P = p_0 \dots p_n$ be a shortest v–u path in G, in particular, $\|P\| = n \leqslant k$. By Lemma 3.2, the subgraph $H_P := H[\{h \in H \mid V(P) \cap V_h \neq \emptyset\}]$ of H is connected. Since $(H, \mathcal{V})$ has (inner-)radial spread r, the subgraph H_P has diameter at most $2r \cdot \|P\| \leqslant 2rk$. As P starts in v and ends in u, the subgraph H_P includes H_v and H_u. Let h' be the node of H witnessing that H_v has radius at most r (in H_v); in particular $h' \in H_v \subseteq H_P$. Since also $h \in H_u \subseteq H_P$ and H_P is connected, there is a path Q from h to h' in H_P of length at most $2rk$. As $Q \subseteq H_P$ and every bag V_g for $g \in H_P$ contains a vertex w of P, which implies $d_G(w, v) \leqslant \|P\| \leqslant k$, it follows by the definition of the new bags V'_h that $Q \subseteq H_P \subseteq H'_v := H[\{h \in H \mid v \in V'_h\}]$. Hence, h' witnesses that H'_v has radius at most $2rk$, so $(H, \mathcal{V}')$ has (inner-)radial spread at most $2rk$. □

3.3. Interplay with quasi-isometries

In this section, we describe the interplay between quasi-isometries and honest graph-decomposition of bounded radial width and spread, i.e. we prove Proposition 1.2. For this, let us recall the definition of quasi-isometries in the case of graphs.

Definition 3.7 (Quasi-isometry). — *For given integers $m, M \geqslant 1$ and $a, A, r \geqslant 0$, an* (m, a, M, A, r)-quasi-isometry *from a graph H to a graph G is a map $\varphi\colon V(H) \to V(G)$ such that*

(Q1) $d_H(h, h') \leqslant m \cdot d_G(\varphi(h), \varphi(h')) + a$ for every $h, h' \in V(H)$,

(Q2) $d_G(\varphi(h), \varphi(h')) \leqslant M \cdot d_H(h, h') + A$ for every $h, h' \in V(H)$, and

(Q3) for every vertex $v \in V(G)$, there exists a node $h \in V(H)$ with $d_G(v, \varphi(h)) \leqslant r$.

Usually (cf. [18, Section 2.1]), quasi-isometries are denoted with only two parameters $L \geqslant 1$ and $C \geqslant 0$: an (L, C)-*quasi-isometry* is precisely an (L, LC, L, C, C)-quasi-isometry. Here, we use a more detailed set of parameters that allows us to describe in a more precise way how graph-decompositions and quasi-isometries interact.

The following is a well-known fact.

Lemma 3.8. — *If a graph H is (L, C)-quasi-isometric to a graph G, then G is $(L, 3LC)$-quasi-isometric to H.*

We split the proof of Proposition 1.2 into the following two lemmata.

Lemma 3.9. — *Let G be a graph, and let $(H, \mathcal{V})$ be an honest graph-decomposition of G of outer-radial width r_0 and outer-radial spread r_1. Then there is a $(2r_1, 2r_1, 2r_0, 0, r_0)$-quasi-isometry from H to G.*

Proof. — Since $(H, (V_h)_{h\in H})$ has outer-radial width r_0, there exists for every bag V_h a vertex $\varphi(h) \in G$ with $V_h \subseteq B_G(\varphi(h), r_0)$. We show that the respective map $\varphi : V(H) \to V(G)$ is the desired quasi-isometry from H to G. Note that the definition of φ immediately implies (Q3) by (H1).

For (Q2) we have to show that for every $h, h' \in V(H)$, we have

$$d_G(\varphi(h), \varphi(h')) \leqslant 2r_0 \cdot d_H(h, h').$$

So let h and h' be arbitrary vertices of H, and let $h_0h_1 \dots h_\ell$ be a shortest h–h' path in H. We now aim to build from this path a $\varphi(h)$–$\varphi(h')$ walk in G of length at most $2r_0\ell$. For every $i \in \{1, \dots, \ell\}$, we fix a vertex $v_i \in V_{h_{i-1}} \cap V_{h_i}$; such v_i exist because the considered H-decomposition of G is honest by assumption. Furthermore, we set $v_0 := \varphi(h)$ and $v_{\ell+1} := \varphi(h')$. Now for every $i \in \{0, \dots, \ell\}$, let P_i be a shortest v_i–v_{i+1} path in G. Since our H-decomposition of G has outer-radial width r_0, the paths P_0 and P_ℓ have length at most r_0, and all other P_i have length at most $2r_0$. Thus, $v_0P_0v_1 \dots v_\ell P_\ell v_{\ell+1}$ is a $\varphi(h)$–$\varphi(h')$ walk in G of length at most $r_0 + 2r_0(\ell - 1) + r_0$, as desired.

For (Q1), we have to check that for every $h, h' \in V(H)$,

$$d_H(h, h') \leqslant 2r_1 \cdot d_G(\varphi(h), \varphi(h')) + 2r_1.$$

So let h and h' be arbitrary vertices of H and let $v_0v_1 \dots v_\ell$ be a shortest $\varphi(h)$–$\varphi(h')$ path in G. Similar as before, we build a h–h' walk in H of length at most $2r_1\ell + 2r_1$. For every $i \in \{1, \dots, \ell\}$ we fix a node $h_i \in V(H)$ satisfying $v_{i-1}v_i \in V_{h_i}$; such h_i exist by (H1). Furthermore, we set $h_0 := h$ and $h_{\ell+1} := h'$. For every $i \in \{0, \dots, \ell\}$, let P_i be a shortest h_i–h_{i+1} path in H. As $h_i, h_{i+1} \in V(H_{v_i})$ for all $i \in \{0, \dots, \ell\}$ and $(H, \mathcal{V})$ has outer-radial spread r_1, all P_i have length at most $2r_1$. Thus, $h_0P_0h_1 \dots h_\ell P_\ell h_{\ell+1}$ is an h–h' walk in H of length at most $2r_1(\ell + 1)$, as desired. □

Lemma 3.10. — *Let φ be an (m, a, M, A, r)-quasi-isometry from a graph H to a graph G, and let $r' \geqslant r$ be an integer. Write $B_h := B_G(\varphi(h), r')$ and set*

$$V_h := \bigcup_{h' \in B_H(h, mr' + \lceil (m+a)/2 \rceil)} B_{h'}.$$

Then $(H, (V_h)_{h\in H})$ is an honest graph-decomposition of G of outer-radial width at most $r' + M(mr' + \lceil (m + a)/2 \rceil) + A$ and (inner-)radial spread at most $4mr' + m + 2a + 1$. Moreover, if $r' \geqslant M + A$, then even the (inner-)radial width is at most $4mr' + m + 2a + 1$.

Proof. — We first show that $(H, (V_h)_{h\in H})$ is indeed an H-decomposition of G'. It then follows immediately from the definition of the V_h that $(H, \mathcal{V})$

is honest (as $\varphi(h) \in V_h$ for all $h \in V(H)$ and $\varphi(h), \varphi(h') \in V_h \cap V_{h'}$ for all $hh' \in E(H)$ because $m \geqslant 1$).

(H1). — By (Q3), each vertex of G is contained in some B_h. Now consider an edge $e = v_0v_1$ of G. There are nodes h_0 and h_1 of H such that $v_0 \in B_{h_0}$ and $v_1 \in B_{h_1}$. Hence, $d_G(\varphi(h_0), \varphi(h_1))$ is at most $2r' + 1$. By (Q1), there exists a path P in H of length at most $m(2r'+1)+a$. There exists a "middle" vertex h on this path P such that both h_0 and h_1 are contained in $B_H(h, mr' + \lceil (m+a)/2 \rceil)$, so both v_0 and v_1 are contained in $B_h \subseteq V_h$. In particular, $e = v_0v_1 \in G[V_h]$, as desired.

(H2). — Let v be any vertex of G, and let h_0 and h_1 be nodes of H such that $v \in V_{h_0} \cap V_{h_1}$. We want to find an h_0–h_1 path P in H such that V_h contains v for every node $h \in P$. By definition of the bags V_h, there are $h_0' \in B_H(h_0, mr' + \lceil (m+a)/2 \rceil)$ and $h_1' \in B_H(h_1, mr' + \lceil (m+a)/2 \rceil)$ such that $v \in B_{h_0'} \cap B_{h_1'}$. Hence, $d_G(\varphi(h_0'), \varphi(h_1'))$ is at most $2r'$. So by (Q1), there is an h_0'–h_1' path P' in H of length at most $2mr' + a$. Let W be the walk $P_0P'P_1$ in H joining h_0 and h_1 where P_0 is a shortest h_0–h_0' path in H and P_1 a shortest h_1'–h_1 path in H. It follows directly from the construction of W that h_0' or h_1' is contained in $B_H(h, mr' + \lceil (m+a)/2 \rceil)$ for every node h visited by W. Since $v \in B_{h_0'} \cap B_{h_1'}$, this implies $v \in V_h$ for every node h visited by W by the definition of V_h. In particular, W contains a h_0–h_1 path P which is as desired.

Secondly, let us verify that $(H, (V_h)_{h \in H})$ has the desired radial spread and outer-radial width. For the radial spread, observe that the above constructed walk W has length at most $2(mr' + \lceil (m+a)/2 \rceil) + 2mr' + a \leqslant 2mr' + m + a + 1 + 2mr' + a = 4mr' + m + 2a + 1$; so the radial spread of $(H, (V_h)_{h \in H})$ is as desired.

For the outer-radial width, consider any node $h \in H$ and vertex $v \in V_h$. By definition, there is $h' \in B_H(h, mr' + \lceil (m+a)/2 \rceil)$ such that $v \in B_{h'}$. By (Q2), we obtain

$$\begin{aligned} d_G(v, \varphi(h)) &\leqslant d_G(v, \varphi(h')) + d_G(\varphi(h), \varphi(h')) \\ &\leqslant r' + M(mr' + \lceil (m+a)/2 \rceil) + A. \end{aligned}$$

Thus, every $v \in V_h$ has distance at most $r' + M(mr' + \lceil (m+a)/2 \rceil) + A$ from $\varphi(h)$. This yields the desired outer-radial width.

To obtain the moreover-part, let us investigate the above equation in more detail. Any shortest v–$\varphi(h')$ in G is contained in $B_h' \subseteq V_h$. Fix a shortest h'–h path Q and replace each edge h_0h_1 of Q by a shortest $\varphi(h_0)$–$\varphi(h_1)$ path $Q_{h_0h_1}$ in G to obtain a $\varphi(h)$–$\varphi(h')$ walk Q' in G. If $r' \geqslant M + A$, the path $Q_{h_0h_1}$ is contained in $B'_{h_0} \subseteq V_h$. Thus, the (inner-)radial width is already at most $r' + M(mr' + \lceil (m+a)/2 \rceil) + A$. □

Proof of Proposition 1.2. — Use Lemma 3.9 for (1) and Lemma 3.10 for (2). □

To conclude this section, let us look at a possible path-way towards omitting the condition on the radial spread. Berger and Seymour proved that a graph has bounded radial tree-width if and only if it is quasi-isometric to a tree [8, 4.1]. More precisely, they show that if G has a T-decomposition of low radial-width for some tree T, then G is quasi-isometric to some tree T' that is obtained from a subtree of T by contracting and subdividing edges.

We ask whether such an argument transfers to arbitrary decomposition graphs:

Question 3.11. — *Given an integer $r \geqslant 1$, does there exist an integer R such that if a graph G has a decomposition modelled on a graph H of radial width at most r, then there exists an honest decomposition of G modelled on a graph H' obtained from a subgraph of H by subdividing and contracting edges such that both its radial width and radial spread are at most R?*

An affirmative answer to Question 3.11 would in particular imply the equivalence of small radial $\mathcal{H}$-width and quasi-isometry to an element of $\mathcal{H}$ for graph classes $\mathcal{H}$ closed under takings subgraphs, and contracting and subdividing edges.

3.4. Quasi-geodesic topological minors

In this section, we study quasi-geodesic topological minors as obstructions to small radial width. We show in Lemma 3.14 that quasi-geodesic topological minors are also an instance of the more general "fat minors" used in Conjecture 1.1. Fat minors are obstructions to small radial width (Lemma 3.15), which then implies that quasi-geodesic topological minors are also such obstructions (Proposition 3.16).

Let us recall the definition of ($\geqslant k$)-subdivisions T_kX and quasi-geodesity from the introduction.

Definition 3.12 (T_kX). — *A* ($\geqslant k$)-subdivision *of a graph X, which we denote with T_kX, is a graph which arises from X by subdividing every edge at least k times, i.e. replacing every edge in X with a new path of length at least $k+1$ such that no new path has an inner vertex in $V(X)$ or on any other new path.*

The original vertices of X are the *branch vertices of the* T_kX. Note that the well-known topological minor relation can be phrased in terms of ($\geqslant 0$)-subdivisions in that a graph X is a *topological minor* of a graph G if G contains a T_0X as a subgraph.

DEFINITION 3.13 (Quasi-geodesic). — *A subgraph X of a graph G is* c-quasi-geodesic (in G) *for some $c \in \mathbb{N}$ if for every two vertices $u, v \in V(X)$ we have $d_G(u,v) \leqslant c \cdot d_X(u,v)$. We call X* geodesic *if it is 1-quasi-geodesic.*

Let us now turn to the question how quasi-geodesic topological minors relate to fat minors, the obstruction investigated by Georgakopoulos and Papasoglu in Conjecture 1.1. For this, let us first recall the definition of "fat minors". Let G, X be graphs. A *model* $(\mathcal{V}, \mathcal{E})$ of X in G is a collection $\mathcal{V}$ of disjoint sets $V_x \subseteq V(G)$ for vertices x of X such that each $G[V_x]$ is connected, and a collection $\mathcal{E}$ of internally disjoint V_{x_0}–V_{x_1} paths E_e for edges $e = x_0x_1$ of X which are disjoint from every V_x with $x \neq x_0, x_1$.[(6)] The V_x are its *branch sets* and the E_e are its *branch paths*. Then X is a *minor* of G if G contains a model of X.

A model $(\mathcal{V}, \mathcal{E})$ of X in G is *K-fat* for $K \in \mathbb{N}$ if $d_G(Y,Z) \geqslant K$ for every two distinct $Y, Z \in \mathcal{V} \cup \mathcal{E}$ unless $Y = E_e$ and $Z = V_x$ for some vertex $x \in V(X)$ incident to $e \in E(X)$, or vice versa. Then X is a *K-fat minor* of G if G contains a K-fat model of X. We remark that the 0-fat minors of G are precisely its minors.

LEMMA 3.14. — *If a graph G contains a T_kX as a c-quasi-geodesic subgraph for some graph X and $c, k \in \mathbb{N}$ with $k \geqslant 2$, then X is a K-fat minor in G for $K = \lfloor \frac{k-2}{2c} \rfloor$.*

Proof. — Denote the c-quasi-geodesic subgraph of G which is a T_kX by X_G. By definition, any two branch vertices of X_G have distance at least $k+1$ in X_G. For $x \in V(X)$, we then choose $V_x = B_{X_G}(x, \lceil (k+1)/4 \rceil)$, and for $e = xx' \in E(X)$, we let E_e be the (unique) V_x-V_x' path in X_G not meeting any other branch vertices; in particular, each E_e has length at least $(k+1) - 2\lceil (k+1)/4 \rceil \geqslant k/2 - 1$. We then let $\mathcal{V}$ be the set of all V_x and $\mathcal{E}$ be the set of all E_e. Since X_G is a T_kX, this construction directly yields $d_{X_G}(Y,Z) \geqslant k/2-1$ for every two distinct $Y, Z \in \mathcal{V} \cup \mathcal{E}$ unless $Y = E_e$ and $Z = V_x$ for some vertex $x \in V(X)$ incident to $e \in E(X)$, or vice versa. Since X_G is a c-quasi-geodesic subgraph of X, we then obtain $d_G(Y,Z) \geqslant d_{X_G}(Y,Z)/c \geqslant \frac{k-2}{2c}$; so X_G is indeed a K-fat model of X in G. □

(6) Note that we here deviate from the definition of model from [11]. However, by enlarging the branch sets V_x along the "adjacent" branch paths E_{xy}, we obtain that this notion of model is equivalent to the notion of model from [11].

Lemma 3.15. — *Suppose X is a K-fat minor in a graph G for some $K \in \mathbb{N}$. If $(H, \mathcal{V})$ is a graph-decomposition of G of radial width less than $K/2$, then X is a minor of H.*

Proof. — Fix a K-fat model $(\mathcal{U}, \mathcal{E})$ of X in G. We aim to define a model $(\mathcal{U}', \mathcal{E}')$ of X in H. For every vertex $x \in X$, let U'_x be the set of nodes $h \in H$ whose corresponding bag V_h meets U_x. Note that the U'_x are pairwise disjoint, as the radial width of $(H, \mathcal{V})$ is $< K/2$ and $(\mathcal{U}, \mathcal{E})$ is K-fat. Since the $G[U_x]$ are connected, the $H[U'_x]$ are connected by Lemma 3.2.

Analogously, for each edge xy of X, the subgraph induced on the nodes $h \in H$ whose V_h meets E_{xy} is connected by Lemma 3.2, meets U'_x and U'_y, and all these subgraphs are pairwise disjoint. Thus, we may pick a U'_x–U'_y path E'_{xy} in each of these subgraphs to obtain a model of X in H. □

Combining the previous two lemmata, we obtain that quasi-geodesic topological minors are indeed obstructions to small radial width:

Proposition 3.16. — *If a graph G contains a $T_{4ck+2}X$ as a c-quasi-geodesic subgraph for some graph X and $c, k \in \mathbb{N}$ with $k \geqslant 2$, then G admits no H-decomposition of radial width less than k modelled on a graph H with no X minor.* □

4. Radial path-width

In this section we prove Theorem 2, which we restate here for convenience.

Theorem 2. — *Let $k \in \mathbb{N}$. If a connected graph G contains no $T_{4k+1}K^3$ as a geodesic subgraph and no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a path P of radial width at most $18k + 2$ and radial spread at most $18k + 1$.*

Moreover, P is $(1, 18k + 2)$-quasi-isometric to G.

We remark that we did not optimise the bounds on the radial width and radial spread.

In fact, we show the following stronger statement, which immediately implies Section 4.

Theorem 4.1. — *Let $k \in \mathbb{N}$, and let P be a longest geodesic path in a connected graph G. If G contains no $T_{4k+1}K^3$ as a geodesic subgraph and no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph, then either P has length at most $18k + 2$ or every vertex of G has distance at most $9k$ from P.*

Let us first show that Theorem 4.1 indeed implies Section 4. For this, we need the following auxiliary lemma, which asserts that taking balls of radius $r \in \mathbb{N}$ around a quasi-geodesic subgraph H of a graph G yields a "partial decomposition" of G modelled on H (see Figure 4.1).

Lemma 4.2. — *Let G be a graph, and let H be a c-quasi-geodesic subgraph of G for some $c \in \mathbb{N}$. Given $r \in \mathbb{N}$ we write $B_h := B_G(h, r)$ for $h \in H$ and set*

$$V_h := \bigcup_{h' \in B_H(h, cr + \lceil c/2 \rceil)} B_{h'}.$$

Then $(H, (V_h)_{h \in H})$ is an honest H-decomposition of $G' := G[\bigcup_{h \in H} B_h]$ of radial width at most $(c+1)r + \lceil c/2 \rceil$ and radial spread at most $4cr + c + 1$.

Moreover, if H is a path, then $(H, (V_h)_{h \in H})$ has radial spread at most $(c+1)r + \lceil c/2 \rceil$.

Proof. — The pair $(H, (V_h)_{h \in H})$ is an honest graph-decomposition of G by Lemma 3.10, as the embedding φ of a c-quasi-geodesic subgraph H of G into $G[\bigcup_{h \in H} B_G(h, r)]$ is a $(c, 0, 1, 0, r)$-quasi-isometry. By Lemma 3.10, this graph-decomposition has (inner-)radial width $r + (cr + \lceil c/2 \rceil) = (c+1)r + \lceil c/2 \rceil$ and radial spread $4rc + c + 1$.

For the "moreover"-part, assume that $H = h_0 \dots h_n$ is a path, and let v be any vertex of G. Let $i, j \in [n]$ be the smallest and largest integer such that $v \in B_{h_k} = B_G(h_k, r)$ for $k = i, j$. Then $d_G(h_i, h_j) \leqslant 2r$, and hence $j - i \leqslant 2cr$ since H is c-quasi-geodesic in G. By the definition of the V_h, it follows that only V_{h_k} with $i - cr + \lceil c/2 \rceil \leqslant k \leqslant j + cr + \lceil c/2 \rceil$ contain v, and hence H_v has radius at most $(c+1)r + \lceil c/2 \rceil$ as it is contained in $B_H(h_k, (c+1)r + \lceil c/2 \rceil)$ for $k := \lfloor (j-i)/2 \rfloor$. □

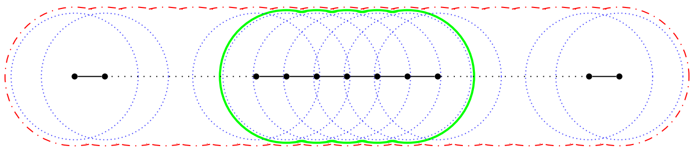

Figure 4.1. A decomposition modelled on the black path using the blue balls as given by Lemma 4.2 for $r = 2$, $c = 1$. The bag corresponding to the centre vertex is green.

Proof of Section 4 given Theorem 4.1. — Let G be a connected graph that contains neither $T_{3k}K_{1,3}$ as a 3-geodesic subgraph nor $T_{4k+1}K^3$ as a geodesic subgraph. Let P' be a longest geodesic path in G. By Theorem 4.1, either P' has length at most $18k + 2$ or every vertex of G has distance at most $9k$ from P'. In the former case, it follows that G has radius at most $18k+2$. Let P be the trivial path on a single vertex p. Then P is $(1, 18k+2)$-quasi-isometric to G, and $(P, (V_p)_{p\in P})$ with $V_p := V(G)$ is an honest decomposition of G of radial width at most $18k+2$ and radial spread 0.

So we may assume the latter case. Set $P := P'$, and apply Lemma 4.2 to $H := P$ and $r := 9k$. This yields an honest P-decomposition of $G' := G[B_G(P, 9k)]$ of radial width at most $18k+1$ and radial spread at most $18k+1$. Since every vertex of G has distance at most $9k$ from P, we have $G' = G$, and hence this is the desired decomposition of G. □

The remainder of this section is devoted to the proof of Theorem 4.1. Let us first give a brief sketch of the proof. For this, let G be a connected graph and, let P be a longest geodesic path in G. For some suitably chosen $r = r(k) < 9k$, we let $G_P := G[B_G(P, r)]$. We then analyse how the components of $G - G_P$ attach to G_P, and show that either all vertices in a component have distance at most $9k$ from P or we can use the component to find a $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph of G or a $T_{3k}K^3$ as a geodesic subgraph of G.

The analysis of the components will be done in three lemmata, Lemmata 4.3, 4.4 and 4.9 below. They are stated in a slightly more general form than needed for the proof of Theorem 4.1, which enables us to use them later also in the proofs of Theorems 3 and 4.

The first lemma, Lemma 4.3 shows that enlarging a c-quasi-geodesic subgraph with a shortest path to it yields a subgraph which is $(2c+1)$-quasi-geodesic. This lets us find a 3-quasi-geodesic $T_{3k}K_{1,3}$ in G if some component of $G - G_P$ attaches "to the middle" of G_P, that is, to some ball $B_G(p, r)$ where p lies "in the middle" of P. The second lemma, Lemma 4.4, demonstrates that we can find a long geodesic cycle in G if some component of $G - G_P$ attaches to G' close to the start and the end of P, but nowhere in between (see Figure 4.2). The third lemma, Lemma 4.9, shows that if a component of $G - G_P$ attaches to G_P only towards one end of P, then its vertices all have distance at most $9k$ from P.

Lemma 4.3. — *Let G be a graph, and let X be a c-quasi-geodesic subgraph of G for some $c \in \mathbb{N}$. If P is a shortest v–X path in G for some vertex $v \in G$, then $X \cup P$ is $(2c+1)$-quasi-geodesic in G.*

Proof. — We have to show that, for every two vertices u and w of $X \cup P$, the distance of u and w in $X \cup P$ is at most $(2c+1)$ times their distance in G. Since X is c-quasi-geodesic in G and P is geodesic in G, it is (by symmetry) enough to consider the case where u is a vertex of P and w is a vertex of X.

Let x be the endvertex of P in X. Since P is a shortest v–X path in G, uPx is a shortest u–X path in G and hence $d_P(u,x) = d_G(u,x) \leqslant d_G(u,w)$ as $w \in X$. We then have $d_X(x,w) \leqslant c \cdot d_G(x,w) \leqslant c \cdot (d_G(x,u) + d_G(u,w))$, where the first inequality follows from X being c-quasi-geodesic in G while the second one applies the triangle inequality. Again using the triangle inequality, we can then combine these inequalities to

$$\begin{aligned} d_{X\cup P}(u,w) &\leqslant d_{X\cup P}(u,x) + d_{X\cup P}(x,w) = d_P(u,x) + d_X(x,w) \\ &\leqslant d_G(u,x) + c \cdot (d_G(x,u) + d_G(u,w)) \leqslant (2c+1) \cdot d_G(u,w), \end{aligned}$$

which shows the claim. □

Lemma 4.4. — *Let $r, n, m_0, m_1 \in \mathbb{N}$ such that $m := n - m_0 - m_1 - 2r > 0$. Let G be a graph containing a geodesic path $P = p_0 \dots p_n$ of length n, and write $B_i := B_G(p_i, r)$ for every $p_i \in P$. Suppose that a component C of $G - \bigcup_{i=0}^{n} B_i$ has at least one neighbour in both $\bigcup_{i=0}^{m_0} B_i$ and $\bigcup_{i=n-m_1}^{n} B_i$, but no neighbours in $\bigcup_{i=m_0+1}^{n-m_1-1} B_i$. Then G contains a geodesic cycle of length at least $2m$.*

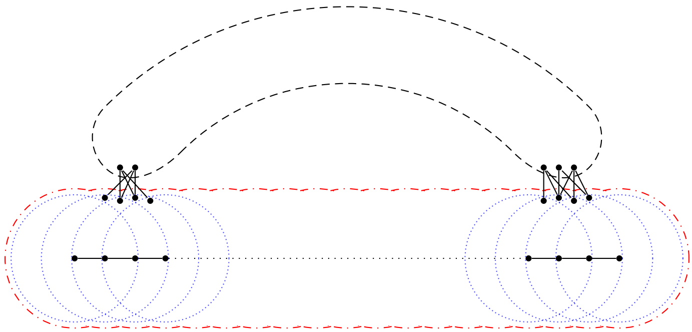

Figure 4.2. The setting of Lemma 4.4.

See Figure 4.2 for a sketch of the situation in Lemma 4.4.

The proof of Lemma 4.4 builds on the study of how cycles interact with a given separation of the graph. More formally, we have the following preparatory lemma.

Lemma 4.5. — *Let G be a graph, S a set of vertices and C a component of $G - S$, and $\{M_0, M_1\}$ be a bipartition of S. Given a cycle O in G, let $W(O)$ be the number of M_0–M_1 paths in O meeting C.*

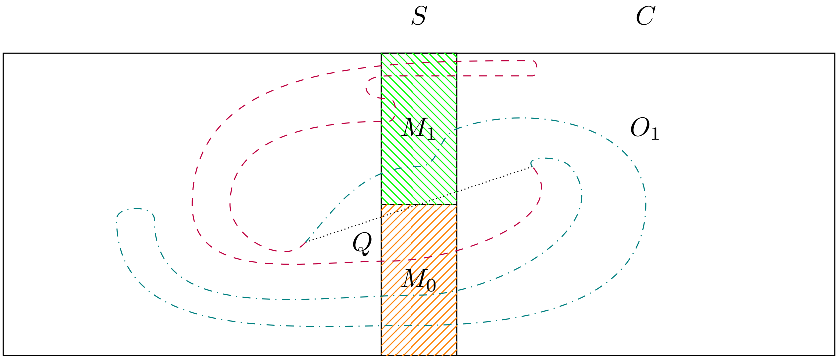

Figure 4.3. The setting of Lemma 4.5. Here, O_2 is the cycle containing Q (black, dotted line) and the purple, dashed part of O_1, and O_3 is the cycle containing Q and the teal, dash dotted part of O_1.

Suppose that Q is an O_1-path of a cycle O_1 in G, and let O_2 and O_3 denote the two cycles in $O_1 \cup Q$ containing Q. Then $W(O_1) + W(O_2) + W(O_3)$ is even.

Proof. — The reader may look at Figure 4.3 to follow the proof more easily. Let u and v be the endvertices of Q on O_1. Write $P_1 := Q$, and let P_2 and P_3 be the two u–v paths in O_1. Then the cycle O_1 consists of P_2 and P_3, and without loss of generality, we may assume that O_2 consists of P_1 and P_2 while O_3 consists of P_1 and P_3.

Let $\mathcal{P}$ be the set of M_0–M_1 paths in G meeting C that are paths in at least one O_i. We remark that every path in $\mathcal{P}$ has all its inner vertices in C since $S = M_0 \cup M_1$ separates C from the rest of the graph. Moreover, let us emphasise that, in the definition of $W(O)$, we do not consider the direction in which O traverses the M_0–M_1 paths it contains.

If $T \in \mathcal{P}$ is a subpath of some P_i, then T is also a subpath of exactly two of the three cycles O_1, O_2 and O_3 and thus contributes exactly 2 to the sum $W(O_1) + W(O_2) + W(O_3)$. So when we check that this sum is even, all the elements of $\mathcal{P}$ that are a subpath of some P_i contribute an even amount to the sum. In particular, if all elements of $\mathcal{P}$ are paths in some P_i, then the claim holds.

So let $\mathcal{P}' \subseteq \mathcal{P}$ consist of precisely those elements of $\mathcal{P}$ that are not a path in any P_i, and suppose that $\mathcal{P}'$ is non-empty. This implies that at least one P_i meets $M_0 \cup M_1$. Every path in $\mathcal{P}$ is a subpath of one of the cycles O_i, which in turn is disjoint from the interior of some P_i. Hence every path in $\mathcal{P}'$ meets the interior of at most, and thus precisely, two of the P_i.

We remark that a path in an O_i that is also a path in some O_j with $i \neq j$ is already a path in some P_i. Thus, every element of $\mathcal{P}'$ contributes exactly 1 to the sum $W(O_1) + W(O_2) + W(O_3)$. In order to prove that this sum is even, it thus suffices to show that the number of paths in $\mathcal{P}'$ is even. We will check this by a case distinction. Note that a path in some O_j is also a path in some P_i if and only if it contains neither of u and v as an inner vertex. Hence, all paths in $\mathcal{P}'$ contain at least one of u and v as an inner vertex. Because $\mathcal{P}'$ is non-empty and all inner vertices of its elements are contained in C, at least one of u and v is contained in C.

First, consider the case that all three paths P_1, P_2 and P_3 meet $M_0 \cup M_1$. In particular, this implies that no path in $\mathcal{P}'$ contains both u and v as an inner vertex. If all paths P_1, P_2 and P_3 have their first vertex in $M_0 \cup M_1$ contained in the same element of $\{M_0, M_1\}$, then no element of $\mathcal{P}'$ contains u as an inner vertex. Otherwise, precisely two paths P_i have their first vertex in $M_0 \cup M_1$ contained in the same element of $\{M_0, M_1\}$. In this case, there are exactly two elements of $\mathcal{P}'$ that contain u as an inner vertex. By symmetry, there are also exactly 2 or 0 elements of $\mathcal{P}'$ that contain v as an inner vertex. Hence, $\mathcal{P}'$ contains precisely $0, 2$ or 4 paths, and thus in this case $W(O_1) + W(O_2) + W(O_3)$ is even.

Secondly, consider the case that precisely one P_i, say P_1, meets $M_0 \cup M_1$. Since P_2 does not meet $M_0 \cup M_1$ and at least one of u and v is contained in C, this implies that both u and v are contained in C. Because $\mathcal{P}'$ is non-empty, the first and last vertex of P_1 in $M_0 \cup M_1$ are contained in distinct elements of $\{M_0, M_1\}$. Thus, $\mathcal{P}'$ contains precisely two paths, one containing P_2 and one containing P_3. Hence, $W(O_1) + W(O_2) + W(O_3)$ is even.

Lastly, consider the case that precisely two P_i meet $M_0 \cup M_1$. Say P_3 does not meet $M_0 \cup M_1$. Thus, again both u and v are contained in C. Here, we need to distinguish some more cases.

The first case which we consider is that both P_1 and P_2 have their first and last vertex in $M_0 \cup M_1$ contained in distinct elements of $\{M_0, M_1\}$ and that the first vertex of P_1 in $M_0 \cup M_1$ and the first vertex of P_2 in $M_0 \cup M_1$ are contained in distinct elements of $\{M_0, M_1\}$. Then $\mathcal{P}'$ contains precisely four elements: two containing P_3 and two not containing any edges of P_3. So again $W(O_1) + W(O_2) + W(O_3)$ is even.

Next, we assume that both P_1 and P_2 have their first and last vertex in $M_0 \cup M_1$ contained in distinct elements of $\{M_0, M_1\}$ but that the first vertex of P_1 in $M_0 \cup M_1$ and the first vertex of P_2 in $M_0 \cup M_1$ are contained in the same element of $\{M_0, M_1\}$. Then $\mathcal{P}'$ contains precisely two elements: both contain P_3. So again $W(O_1) + W(O_2) + W(O_3)$ is even.

Now we assume that at least one of P_1 and P_2, say P_1, has their first and last vertex in $M_0 \cup M_1$ contained in the same element of $\{M_0, M_1\}$, say in M_0. Since $\mathcal{P}'$ is non-empty and every path in $\mathcal{P}'$ contains u or v as inner vertex, this implies that either the first or the last vertex of P_2 in $M_0 \cup M_1$ is contained in M_1. If only the first or only the last vertex of P_2 in $M_0 \cup M_1$ is contained in M_1, then $\mathcal{P}'$ contains precisely two elements: one of these contains P_3 and the other does not contain edges of P_3. If both the first and the last vertex of P_2 in $M_0 \cup M_1$ are contained in M_1, then $\mathcal{P}'$ contains also precisely two elements: neither of them contains edges of P_3. Again in both cases, $W(O_1) + W(O_2) + W(O_3)$ is even. □

Proof of Lemma 4.4. — Note that, as $n > m_0 + m_1$, we in particular have $n - m_1 > m_0$. We set

$$M_0 := N_G(C) \cap \bigcup_{i=0}^{m_0} B_i \text{ and } M_1 := N_G(C) \cap \bigcup_{i=n-m_1}^{n} B_i.$$

As C has no neighbours in $\bigcup_{i=m_0+1}^{n-m_1-1} B_i$, the neighbourhood of C is equal to $M_0 \cup M_1$ and C is a component of $G - M_0 - M_1$.

First, we note that M_0 and M_1 are disjoint. Indeed, if $u \in B_{j_0}$ for some $j_0 \leqslant m_0$ and $v \in B_{j_1}$ for some $j_1 \geqslant n - m_1$, then using the triangle inequality we obtain

$$\begin{aligned} n - m_0 - m_1 &\leqslant d_P(p_{j_0}, p_{j_1}) = d_G(p_{j_0}, p_{j_1}) \\ &\leqslant d_G(p_{j_0}, u) + d_G(u, v) + d_G(v, p_{j_1}) = d_G(u, v) + 2r. \end{aligned} \tag{4.1}$$

So $d_G(u, v) \geqslant n - m_0 - m_1 - 2r = m > 0$ by assumption. In particular, u and v are distinct vertices, and thus M_0 and M_1 are disjoint.

We now define, for a cycle O in G, the number $W(O)$ as the number of M_0–M_1 paths in O which have an inner vertex in C.

The remainder of the proof now consists of three steps. First, we show that a cycle O with $W(O) \neq 0$ has length at least $2m$. Second, we find a cycle O with odd $W(O)$, and lastly we show that a shortest such cycle is geodesic in G.

Claim 4.6. — *Each cycle O in G with $W(O) \neq 0$ has length at least $2m$.*

Proof. Since $W(O) \neq 0$, the cycle O contains at least one M_0–M_1 path. As the number of M_0–M_1 paths is even for every cycle in G, the cycle O contains at least two M_0–M_1 paths, which then have to be internally disjoint. By (4.1), every such path has length at least m, and thus O has length at least $2m$. ■

Claim 4.7. — *There exists a cycle O in G with odd $W(O)$.*

Proof. Among all M_0–M_1 paths that contain an inner vertex in C, let Q be a shortest such path, and denote by v_0 and v_1 its end vertices in M_0 and M_1, respectively. Let R_0 be a shortest v_0–P path in G and R_1 a shortest v_1–P path. Since $v_i \in M_0 \cup M_1 = N_G(C)$, both R_i have length exactly r. If R_0 and R_1 share a vertex x, say such that R_0x is at least as long as R_1x, then R_1xR_0 contains a path of length at most r from v_1 to some p_s with $s \leqslant m_0$, contradicting $v_1 \in M_1$ and $M_0 \cap M_1 = \emptyset$. Hence, R_0 and R_1 are disjoint. Now the concatenation R_0QR_1 is a path that starts and ends in distinct vertices of P and is internally disjoint from P. Hence, it can be closed by a (unique) subpath of P to a cycle O. By construction of O, there is precisely one M_0–M_1 path in O that has an inner vertex in C, and that is Q. In particular, $W(O)$ is odd. ■

Claim 4.8. — *Let O be a cycle in G with odd $W(O)$. If O is not geodesic, then there is a cycle O' that is shorter than O such that $W(O')$ is odd.*

Proof. Since $O_1 := O$ is not geodesic, there is an O_1-path Q in G with endvertices u and v in O which is shorter than the distance of u and v in O_1. Let O_2 and O_3 be the two cycles in $O_1 \cup Q$ which contain Q. Since both O_2 and O_3 are shorter than O_1, it suffices to show that at least one of $W(O_2)$ and $W(O_3)$ is odd. Now $W(O_1)$ is odd by assumption, so it is enough to show that the sum $W(O_1) + W(O_2) + W(O_3)$ is even. This however follows from Lemma 4.5 applied to the component C together with the bipartition $\{M_0, M_1\}$ of $S = N_G(C)$. ■

To formally complete the proof, pick a cycle O in G with $W(O)$ odd such that O is as short as possible among these. Such O exists by Claim 4.7 and is geodesic in G by Claim 4.8. Moreover, O has length at least $2m$ by Claim 4.6, as desired. □

Lemma 4.9. — *Let U be a set of vertices of a connected graph G, and let $p_0 \in V(G) \setminus U$ have maximal distance from U in G. Let $P = p_0 \dots p_n$ be a shortest p_0–U path in G. Given $r \in \mathbb{N}$, we write $B_i := B_G(p_i, r)$ for every $p_i \in P$. Suppose that some component C of $G - \bigcup_{i=0}^{n} B_i$ is disjoint from U and satisfies $N_G(C) \subseteq \bigcup_{i=0}^{\ell} B_i$ for some $\ell \leqslant n$. Then $d_G(v, P) \leqslant 2r + \ell$ for every $v \in C$.*

Proof. — Let v be an arbitrary vertex of C, and let Q be a shortest v–U path in G with endvertex $u \in U$. As $V(C) \cap U = \emptyset$, the path Q intersects

$\bigcup_{i=0}^{n} B_i$. Let q be the first vertex on Q in $\bigcup_{i=0}^{n} B_i$, and let j be the smallest index of a ball B_j with $q \in B_j$. Note that $j \leqslant \ell$ since $N_G(C) \subseteq \bigcup_{i=0}^{\ell} B_i$.

Since P is a longest geodesic path in G which ends in U, we have that $n = \|P\| \geqslant \|Q\| = d_G(v,u) = d_G(v,q) + d_G(q,u)$, where the last equality follows from Q being geodesic in G. Additionally, since $p_n \in U$, we have $d_G(p_j,p_n) \leqslant d_G(p_j,u) \leqslant d_G(p_j,q) + d_G(q,u) = r + d_G(q,u)$. These two inequalities combine to

$$d_G(v,q) \leqslant n - d_G(q,u) \leqslant n - (d_G(p_j,p_n) - r).$$

All in all, we get

$$\begin{aligned} d_G(v,P) &\leqslant d_G(v,p_j) \leqslant d_G(v,q) + r \\ &\leqslant n - (d_G(p_j,p_n) - r) + r = n - (n-j) + 2r \leqslant 2r + \ell, \end{aligned}$$

where the last inequality follows from $j \leqslant \ell$. This concludes the proof of the claim. □

With all the previous lemmata at hand, we are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. — Let G be a connected graph that contains neither $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph nor $T_{4k+1}K^3$ as a geodesic subgraph. Let $P = p_0 \ldots p_n$ be a longest geodesic path in G. We may assume $n \geqslant 18k+3$; otherwise we are done. For every $0 \leqslant i \leqslant n$, define $B_i := B_G(p_i, 3k)$ as the ball in G of radius $3k$ around p_i. If $G = G_P$, then we are done. Otherwise, we consider the components of $G - G_P$. Since G is connected by assumption, each component C of $G - G_P$ has a neighbour in at least one B_i. The following two claims show that in fact $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$ or $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$.

Claim 4.10. — *No component C of $G - G_P$ has a neighbour in B_i with $3k < i < n-3k$.*

Proof. Suppose for a contradiction that there is such a component C. We show that G then contains a $T_{3k}K_{1,3}$ as 3-quasi-geodesic subgraph, which contradicts our assumption on G.

Let $v \in C$ be a vertex with a neighbour in B_i with $3k < i < n-3k$, and let Q be a shortest v–p_i path in G. The choice of v guarantees that the path Q has length exactly $3k+1$, and thus Q is a shortest v–P path in G as every vertex in $G - G_P$ and hence in C has distance at least $3k+1$ to P. By Lemma 4.3, $P \cup Q$ is a 3-quasi-geodesic subgraph of G. It follows from $3k < i < n-3k$ that $P \cup Q$ is a $T_{3k}K_{1,3}$, as desired. ■

CLAIM 4.11. — *No component C of $G - G_P$ has at least one neighbour in $\bigcup_{i=0}^{3k} B_i$, at least one neighbour in $\bigcup_{i=n-3k}^{n} B_i$ and no neighbours in any B_i with $3k < i < n - 3k$.*

Proof. Suppose for a contradiction that there is such a component C. Applying Lemma 4.4 to C with $m_0 = m_1 = r = 3k$, we find that G contains a geodesic cycle O of length at least $2(n-12k)$. But $n \geqslant 18k+3$ as we assumed the graph to have a large radius, so O has length at least $2(18k+3-12k) = 12k+6$. Thus, O is a geodesic cycle in G of length at least $12k+6$, a contradiction to our assumptions on G. ■

Thus, each component of $G-G_P$ attaches either only to balls B_i with $i \leqslant 3k$ or to balls B_i with $i \geqslant n-3k$. But the vertices in such components have bounded distance from P in G, as the following claim shows.

CLAIM 4.12. — *Let C be a component of $G - G_P$ such that $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$ or $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$. Then every vertex $v \in C$ has distance at most $9k$ from P in G.*

Proof. If $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$, then the claim follows by applying Lemma 4.9 with $U = \{p_n\}$. Otherwise, $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$, and the claim follows by applying Lemma 4.9 with $U = \{p_0\}$. ■

By Claims 4.10–4.12, every vertex of G has distance at most $9k$ from P, as desired. □

5. Radial cycle-width

In this section we build on our result on radial path-width, Section 4, and use a small adaptation of its proof to address radial cycle-width by proving Theorem 3. More precisely, Theorem 3 follows from the lemmata that we have already shown in Section 4. Let us restate Theorem 3 here for convenience.

THEOREM 3. — *Let $k \in \mathbb{N}$. If a connected graph G contains no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a path or cycle C of radial width at most $18k+2$ and radial spread at most $36k+2$.*

Moreover, C is $(1, 18k+2)$-quasi-isometric to G.

We remark that we did not optimise the bound on the radial width and radial spread.

In fact, we show the following stronger statement, which immediately implies Section 5.

Theorem 5.1. — *Let $k \in \mathbb{N}$, and let G be a connected graph. If G contains no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph, then there exists a geodesic cycle or path C in G such that C is either a path of length at most $18k+2$ or every vertex of G has distance at most $9k$ from C.*

Let us first show that Theorem 5.1 implies Section 5.

Proof of Section 5 given Theorem 5.1. — Let G be a connected graph that contains no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph, and let C' be given by applying Theorem 5.1 to G. If C' is a path of length at most $18k+2$, then G has radius at most $18k+2$. Let C be the trivial path on a single vertex c. Then C is $(1, 18k+2)$-quasi-isometric to G, and $(C, (V_c))$ with $V_c := V(G)$ is an honest decomposition of G of radial width at most $18k+2$ and radial spread 0.

So we may assume that every vertex of G has distance at most $9k$ from C. Set $C := C'$, and apply Lemma 4.2 to $H := C$ and $r := 9k$. This yields an honest C-decomposition of $G' := G[B_G(C, 9k)]$ of radial width at most $18k+1$ and radial spread at most $36k+2$. Since every vertex of G has distance at most $9k$ from C, we have $G' = G$, and hence this is the desired decomposition of G. □

Let us now prove Theorem 5.1.

Proof of Theorem 5.1. — Let G be a connected graph that contains no $T_{3k}K_{1,3}$ as a 3-quasi-geodesic subgraph. Applying Theorem 4.1 to G yields that G either contains a geodesic path P such that $C := P$ is as desired, or G contains a $T_{4k+1}K^3$ as a geodesic subgraph. Since we are done in the first case, we may thus assume that G contains a $T_{4k+1}K^3$ as a geodesic subgraph, which means that there exists a geodesic cycle C in G of length at least $3 \cdot (4k+2) = 12k+6$. Set $G_C := G[B_G(C, 3k)]$. If $G = G_C$, then we are done.

Otherwise, there exists a neighbour v of $V(G_C)$ in G since G is connected. Let P be a shortest v–C path in G, and let u be its endvertex in C. Since $v \in N_G(G_C)$, it has distance $3k+1$ from C, and thus P has length $3k+1$. Let $Q := B_C(u, 3k+1)$ be the subpath of C of length $6k+2$ which contains u as its "middle" vertex. Note that Q is actually a path as C has length at least $12k+6 > 6k+2$. Since P is a path from v to C which ends in u, the paths P and Q only meet in u. Hence, $P \cup Q$ is a $T_{3k}K_{1,3}$ in G.

To conclude the proof and obtain the desired contradiction to our assumptions on G, it remains to show that $P \cup Q$ is 3-quasi-geodesic in G. Since Q is a subpath of C with $\|Q\| = 6k+2 \leqslant \frac{1}{2}(12k+6) \leqslant \frac{1}{2}\|C\|$, we have $d_Q(u,v) = d_C(u,v)$ for every two vertices $u, v \in Q$. Thus, Q is geodesic in G, since C is geodesic in G. By Lemma 4.3, $P \cup Q$ then is a 3-quasi-geodesic subgraph of G, as desired. □

6. Radial star-width

In this section we prove Theorem 4, which we restate here for convenience.

THEOREM 4. — *Let $k \in \mathbb{N}$. If a connected graph G contains no $T_k K^3$ as a geodesic subgraph and no $T_{3k} W$ as a 3-quasi-geodesic subgraph, then G admits an honest decomposition modelled on a subdivided star of radial width at most $58k+9$ and radial spread at most $30k+7$. Moreover, there exists some $C_k \in \mathbb{N}$ such that some subdivided star is $(1, C_k)$-quasi-isometric to G.*

Recall that W is the *wrench* graph depicted in Figure 1.1. As we already mentioned in the introduction, one can check by carefully reading the proof that the subdivided star which we construct for the first part of the statement already satisfies the "moreover"-part, and that we may choose $C_k = 60k + 14$ (see the paragraph after the proof of Section 6 for details).

Before we start with the proof of Section 6, we first show an auxiliary lemma. Recall that in the proof of Section 4 we used Lemma 4.4 as a tool to show that a graph contains a geodesic $T_k K^3$. Similarly, we will use the following lemma in the proof of Section 6 to show that a graph contains a 3-quasi-geodesic $T_{3k} W$.

LEMMA 6.1. — *Let $P = p_0 \dots p_n$ be a geodesic path in a graph G. Given two vertices $u, v \in G$, let Q be a shortest u–P path in G with endvertex $q \in P$, and let R be a shortest v–$(P \cup Q)$ path in G with endvertex $r \in P \cup Q$. Suppose that at least one of the following two conditions hold:*

(i) $r \in P$ and $d_G(q, r) \geqslant 4 \max\{d_G(u, P), d_G(v, P)\}$;

(ii) $r \in Q$ and $d_G(q, r) \geqslant 4 \max\{d_G(v, Q), d_G(p_0, q), d_G(p_n, q)\}$.

Then $P \cup Q \cup R$ is a 3-quasi-geodesic subgraph of G.

Proof. — We first consider the case that $r \in P$. By Lemma 4.3, we have that $P \cup Q$ and $P \cup R$ are each 3-quasi-geodesic subgraphs of G. Hence, in order to show that $X := P \cup Q \cup R$ is 3-quasi-geodesic, it suffices to consider vertices $x, y \in X$ with $x \in Q$ and $y \in R$.

Since P is geodesic, we have that $d_X(q,r) = d_G(q,r) \leqslant d_G(q,x) + d_G(x,y) + d_G(y,r)$ and thus

$$\begin{aligned} d_G(x,y) &\geqslant d_G(q,r) - d_G(x,q) - d_G(y,r) \\ &\geqslant 4 \cdot \max\{d_G(u,P), d_G(v,P)\} - d_G(u,P) - d_G(v,P) \\ &\geqslant 2 \cdot \max\{d_G(u,P), d_G(v,P)\} \geqslant 2 \cdot \max\{d_G(x,q), d_G(y,r)\}. \end{aligned}$$

Therefore, making again use of $d_G(q,r) \leqslant d_G(q,x) + d_G(x,y) + d_G(y,r)$, we obtain

$$\begin{aligned} d_X(x,y) &= d_X(x,q) + d_X(q,r) + d_X(r,y) = d_G(x,q) + d_G(q,r) + d_G(r,y) \\ &\leqslant 2 \cdot d_G(x,q) + 2 \cdot d_G(y,r) + d_G(x,y) \leqslant 3 \cdot d_G(x,y). \end{aligned}$$

For the second case, assume that $r \in Q$. Similar as before, we find that $P \cup Q$ and $Q \cup R$ are each 3-quasi-geodesic subgraphs of G. Hence, we only need to consider vertices $x, y \in X$ with $x \in P$ and $y \in R$. As before, we find that

$$\begin{aligned} d_G(x,y) &\geqslant d_G(q,r) - d_G(x,q) - d_G(y,r) \\ &\geqslant 2 \cdot \max\{d_G(v,Q), d_G(p_0,q), d_G(p_n,q)\}, \end{aligned}$$

and thus

$$\begin{aligned} d_X(x,y) &= d_G(x,q) + d_G(q,r) + d_G(r,y) \\ &\leqslant 2 \cdot d_G(x,q) + 2 \cdot d_G(y,r) + d_G(x,y) \leqslant 3 \cdot d_G(x,y). \end{aligned}$$ □

Now we prove Section 6. We remark that we did not optimise the bounds on the radial width and radial spread.

Proof of Section 6. — Let G be a graph that contains neither $T_k K^3$ as a geodesic subgraph nor $T_{3k} W$ as a 3-quasi-geodesic subgraph. We will construct a subdivided star S and an S-decomposition of G of radial width at most $58k + 9$ and radial spread at most $58k + 9$. The "moreover" part of the statement then follows: By Lemmata 3.8 and 3.9, there exists an (L, C)-quasi-isometry from G to S such that L and C depend on k only. Applying [23, 1.3] (to $L, C, 2$)[(7)] then yields a constant C'_k such that G is $(1, C'_k)$-quasi-isometric to some subdivided star S'. By Lemma 3.8, it then follows that S' is $(1, C_k)$-quasi-isometric to G for $C_k := 3C'_k$.

Let $P = p_0 \dots p_n$ be a longest geodesic path in G. Observe that we may assume $n \geqslant 58k + 10$; otherwise, G has a trivial $\mathcal{H}$-decomposition into a single ball of radius at most $58k + 9$. For every $0 \leqslant i \leqslant n$ define B_i as the ball in G of radius $3k$ around p_i, and let $G_P := G[\bigcup_{i=0}^n B_i]$.

(7) Note that we use here that subdivided stars admit path-decompositions into bags of size 3 and thus have path-width 2.

In the following claims we will analyse how the components of $G - G_P$ attach to G_P. Note that, as we assumed G to be connected, every component of $G - G_P$ has some neighbour in some B_i. First we show that we may assume that some component of $G - G_P$ attaches to G_P somewhere in the middle of P.

Claim 6.2. — *Either G has an honest P-decomposition of radial width at most $58k+9$ and radial spread at most $58k+9$ or some vertex of $G-G_P$ has a neighbour in some B_s with $23k+5 \leqslant s \leqslant n-23k-5$.*

Proof. We assume for a contradiction that whenever a component of $G-G_P$ has a neighbour in some B_s then $s \leqslant 23k+4$ or $n-23k-4 \leqslant s$.

First we consider the case that some component C of $G-G_P$ has at least one neighbour in both $\bigcup_{i=0}^{23k+4} B_i$ and $\bigcup_{i=n-23k-4}^{n} B_i$. Let $m_0 = m_1 = 23k+4$ and $r = 3k$. As

$$n-m_0-m_1-2r = n-(23k+4)-(23k+4)-6k = n-52k-8 \geqslant 2k+2 \geqslant 2,$$

we can apply Lemma 4.4 to obtain a geodesic cycle in G of length at least $2\cdot(2k+2) > 3k+3$ which contradicts our assumption that G does not contain a geodesic T_kK^3.

Thus, for every component C of $G-G_P$, either $N_G(C) \subseteq \bigcup_{i=0}^{23k+4} B_i$ or $N_G(C) \subseteq \bigcup_{i=n-23k-4}^{n} B_i$ (this includes in particular the case that $G = G_P$). Then by Lemma 4.9 applied to $U = \{p_0\}$ or $U = \{p_n\}$, every vertex in $G-G_P$ has distance at most $2\cdot 3k+23k+4 = 29k+4$ to P. Hence, by defining $B_i' = B_G(p_i, 29k+4)$, every vertex of G is contained in some B_i'. So by Lemma 4.2 applied to the geodesic path P and to $r = 29k+4$ we obtain an honest P-decomposition of G of radial width at most $2\cdot(29k+4)+1 = 58k+9$ and radial spread at most $2\cdot(29k+4)+1 = 58k+9$. ■

By Claim 6.2 (and since P is a path and hence a subdivided star) we may assume that there is some s with $23k+5 \leqslant s \leqslant n-23k-5$ such that some vertex of $G-G_P$ has a neighbour in B_s. We fix s for the rest of the proof. The following two claims now show that every component of $G-G_P$ can only attach to G_P either close to the start of P or close to the end of P or close to p_s.

Claim 6.3. — *If a vertex of $G-G_P$ has a neighbour in a ball B_t, then $t \leqslant 3k$ or $s-12k-3 \leqslant t \leqslant s+12k+3$ or $n-3k \leqslant t$.*

Proof. We assume for a contradiction that there is some vertex v of $G-G_P$ that has a neighbour in some B_t with $3k < t < s-12k-3$, the other case is symmetric. Let $R = r_0 \dots r_{3k+1}$ be a shortest path from $v = r_0$ to $p_t = r_{3k+1}$ in G. Similarly, let u be a vertex of $G-G_P$ that has a neighbour in B_s and let $Q = q_0 q_1 \dots q_{3k+1}$ be a shortest path from $u = q_0$ to $p_s = q_{3k+1}$ in G. Then $p_t P p_s$ has length at least $12k + 4 = 4 \cdot (3k + 1)$, so by applying Lemma 6.1 to $p_{t-3k-1} P p_{s+3k+1}$, Q and R we obtain a 3-quasi-geodesic $T_{3k} W$ in G, a contradiction to our assumption on G. ■

Claim 6.4. — *For every component C of $G - G_P$, either $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$ or $N_G(C) \subseteq \bigcup_{i=s-12k-3}^{s+12k+3} B_i =: D_s$ or $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$.*

Proof. First, we assume for a contradiction that C has neighbours in both $\bigcup_{i=0}^{3k} B_i$ and D_s. Let $r = 3k$, $m_0 = 3k$ and $m_1 = n - (s - 12k - 3)$. Then, since $s \geqslant 23k + 5$,

$$n - m_0 - m_1 - 2r = n - 3k - (n - (s - 12k - 3)) - 6k = s - 21k - 3 \geqslant 2k + 2 \geqslant 2,$$

so we can apply Lemma 4.4 to obtain a geodesic cycle in G of length at least $2 \cdot (2k + 2) > 3k + 3$. This contradicts our assumption that G does not contain a geodesic $T_k K^3$.

Hence, if C has neighbours in D_s, then it does not have neighbours in $\bigcup_{i=0}^{3k} B_i$, and by symmetry it neither has neighbours in $\bigcup_{i=n-3k}^{n} B_i$. Furthermore, if C has neighbours in both $\bigcup_{i=0}^{3k} B_i$ and $\bigcup_{i=n-3k}^{n} B_i$, then it follows again from Lemma 4.4 that G contains a geodesic $T_k K^3$, which yields the same contradiction. ■

Let us first consider all the components of $G - G_P$ that attach to G_P either close to the start or close to the end of P. By Lemma 4.9 these components can only contain vertices which are close to P. This fact allows us to find a path-decomposition of $G' := G[V']$ of low radial width and spread where V' is the union of the B_i and the vertices of components C of $G - G_P$ with $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$ or $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$, as follows.

Claim 6.5. — *All vertices of G' have distance at most $9k$ from P. Also, there is an honest decomposition $(P, \mathcal{V}')$ of G' of radial width at most $24k + 5$ and radial spread at most $24k + 5$ such that*

- *all vertices in the bag V'_s of the node p_s of P have distance at most $24k + 5$ from p_s in $G[V'_s]$, and*
- *for every $v \in N_G(G')$ the set $N_G(v) \cap V'$ is contained in the bag V'_s.*

Proof. First, we show that vertices of $G' - G_P$ are close to p_0 or p_n. Let C be a component of $G' - G_P$ with $N_G(C) \subseteq \bigcup_{i=0}^{3k} B_i$. By Lemma 4.9 applied to $U = \{p_n\}$ and $\ell = 3k$, every vertex $v \in C$ has distance at most $2 \cdot (3k) + 3k = 9k$ to P, and thus has distance at most $12k$ to p_0. By symmetry, every component C of $G' - G_P$ with $N_G(C) \subseteq \bigcup_{i=n-3k}^{n} B_i$ only contains vertices whose distance from p_n is at most $12k$.

Now we construct the P-decomposition. For $i \in [n]$ define

$$V'_i := \bigcup_{p_j \in B_P(p_i, 12k+3)} B_{G'}(p_j, 12k+2).$$

By Lemma 4.2, the V'_i are the bags of a P-decomposition of radial spread at most $2(12k+2)+1 = 24k+5$, and every element of V'_i has distance at most $24k+5$ from p_i in $G[V'_i]$ by construction. Furthermore, every component of $G' - G_P$ is contained in $B_{G'}(p_0, 12k+2) \subseteq V'_0$ or in $B_{G'}(p_n, 12k+2) \subseteq V'_n$.

Also, for every B_i that contains a neighbour of $G - G' \subseteq G - G_P$ we have $s - 12k - 3 \leqslant i \leqslant s + 12k + 3$ by Claim 6.3 and thus $B_i \subseteq B_{G'}(p_i, 12k+2) \subseteq V'_s$. As V'_s is the bag corresponding to p_s, this completes the proof. ∎

We now construct path-decompositions of low radial width and spread of the remaining components, that is of the components of $G - V'$. We then combine these path-decompositions with the path-decomposition $(P, \mathcal{V}')$ of G' to a star-decomposition of G, whose central node will be p_s. For this, we need to enlarge the bag V'_s assigned to p_s a little. Indeed, at the moment, the components of $G - G'$ need not have low radial path-width; in fact, they can still be star-like. But if we delete larger balls around the nodes in P that are close to p_s, we indeed end up with components that are path-like. For this, we define V'' to be the set of vertices of G that have distance at most $18k+4$ from P, and let $G'' := G[V'']$ (so $G' \subseteq G''$). In particular, by Claim 6.3, every component of $G - G''$ "attaches in the middle of P", i.e. every neighbour of G'' in a component of $G - G''$ has distance $18k+5$ to a p_i with $s - 12k - 3 \leqslant i \leqslant s + 12k + 3$. Let further $\mathcal{C}$ be the set of components that contain a vertex whose distance from P is at least $24k+5$, and let $\mathcal{C}'$ be the set of all other components of $G - V''$. We now add all vertices in $V'' \setminus V'$ and all components from $\mathcal{C}'$ to the bag V'_s that is indexed by p_s. More precisely, we define

$$V_s := V'_s \cup (V'' \setminus V') \cup \{V(C) \colon C \in \mathcal{C}'\}.$$

Further, we set $V_i := V'_i$ for all $i \neq s \in [n]$. Let also $V''' := V'' \cup V_s$ and $G''' := G[V''']$.

Claim 6.6. — *$(P, \mathcal{V})$ is a decomposition of G''' of radial width at most $36k+7$ and radial spread at most $24k+5$ and every vertex in V_s has distance at most $36k+7$ from p_s in $G[V_s]$.*

Proof. Since $N_G(v) \cap V'$ is contained in $V'_s \subseteq V_s$ for every vertex $v \in N_G(G')$ by Claim 6.5, and because $(P, \mathcal{V}')$ is a decomposition of G', it follows that $(P, \mathcal{V})$ is a decomposition of G'''. Its radial spread is at most $24k+5$ by

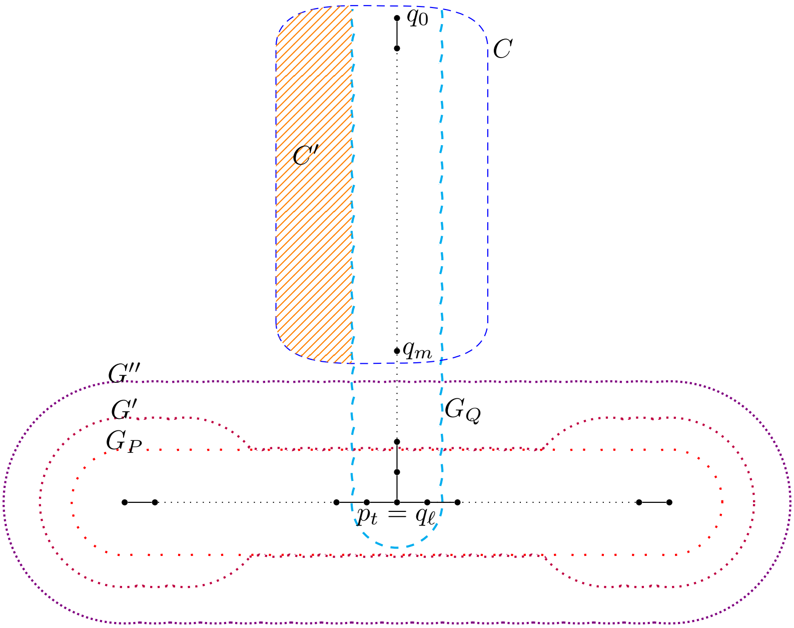

Figure 6.1. The setting of the proof of Claim 6.7.

Claim 6.5 and because each vertex in $G''' - V'$ is only contained in V_s. Moreover, every part $G[V_i]$ with $i \neq s$ has radius at most $24k + 5$ by Claim 6.5, so it remains to consider $G[V_s]$.

Let $v \in V_s$ be given. If $v \in V_s'$, then $d_{G[V_s]}(v, p_s) \leqslant 24k+5$ by Claim 6.5. Otherwise, v has distance at most $24k+4$ from some vertex p_i of P, where $s - 12k - 3 \leqslant i \leqslant s + 12k + 3$ by Claim 6.3, and hence

$$d_{G[V_s]}(v, p_s) \leqslant d_{G[V_s]}(v, p_i) + d_{G[V_s]}(p_i, p_s) \leqslant (24k+4) + (12k+3) = 36k+7$$

where we used that $p_i P p_s \subseteq G[V_s'] \subseteq G[V_s]$ by the definition of V_s'. ■

We now show that all components of $G - V'''$, i.e. those in $\mathcal{C}$, are path-like.

Claim 6.7. — *Let C be a component in $\mathcal{C}$. Then there is an honest decomposition of $G[V(C) \cup V_s]$ modelled on a path Q of radial width at most $54k+8$ and radial spread at most $12k+2$ such that the bag corresponding to the last node of Q contains V_s, and V_s only meets bags assigned to nodes of distance at most $6k$ to the last node of Q.*

Proof. The reader may look at Figure 6.1 to follow the proof more easily. Let q_0 be a vertex of C of maximal distance from P. Since $C \in \mathcal{C}$ the vertex q_0 has distance at least $24k + 5$ from P. Let $Q = q_0 \dots q_\ell$ be a shortest path in G from q_0 to P (so Q is geodesic in G). Note that C is contained in a component of $G - G'$, and thus Q ends in a vertex $p_t = q_\ell$ with $s-12k-3 \leqslant t \leqslant s+12k+3$ by Claim 6.3. In particular $3k < t < n-3k$. Let m be the last index such that $q_m \notin G''$. Then Qq_m is a shortest q_0–$N_G(G'')$-path, and $m = \ell - 18k - 4$.

For every $0 \leqslant i \leqslant \ell$ we define B_i^Q to be the ball in G of radius $3k$ around q_i, and let $G_Q = G[\bigcup_{i=0}^{\ell} B_i^Q]$. We claim that every vertex of $C - G_Q$ has distance at most $12k$ from q_0. To see this, let C' be any component of $C - G_Q$, and let C'' be the component of $G - G_Q$ containing C'. We first show that C'' has no neighbour in $\bigcup_{i=3k+1}^{\ell} B_i^Q$ and that this in particular implies that $C' = C''$.

Towards a contradiction, we first assume that C'' has a neighbour in some B_j^Q with $3k < j < \ell - 12k - 3$. Then q_jQ has length at least $12k+4 = 4 \cdot (3k+1)$. So Lemma 6.1 (ii) applied to $p_{t-3k-1}Pp_{t+3k+1}$, $q_{j-3k-1}Q$ and a shortest path from C'' to q_j yields that G contains a 3-quasi-geodesic $T_{3k}W$, a contradiction.

Second, suppose for a contradiction that C'' has a neighbour in the set $\bigcup_{i=\ell-12k-3}^{\ell} B_i^Q$. In addition, C'' has a neighbour in $\bigcup_{i=0}^{3k} B_i^Q$. Indeed, since vertices in $\bigcup_{i=\ell-12k-3}^{\ell} B_i^Q$ have distance at most $15k+3 < 18k+4$ from P, they are neither contained in C' nor do they have neighbours in C'. Since $C' \subseteq C''$, this implies that C' has no neighbours in $\bigcup_{i=3k+1}^{\ell} B_i^Q$. As C is connected and contains $q_0 \in G_Q$, C' has a neighbour in G_Q which then has to be contained in $\bigcup_{i=0}^{3k} B_i^Q$. Hence, C'' also has a neighbour in $\bigcup_{i=0}^{3k} B_i^Q$.

For $r = 3k$, $m_0 = 3k$ and $m_1 = 12k + 3$, we have that

$$\ell - m_0 - m_1 - 2r = \ell - 3k - 12k - 3 - 6k = \ell - 21k - 3 \geqslant 2k + 2,$$

so we can apply Lemma 4.4 to Q and C'' to obtain a geodesic cycle of length at least $2(2k+2) > 3k+3$, which contradicts our assumption on G. Thus, $N_G(C'') \subseteq \bigcup_{i=0}^{3k} B_i^Q$. As G'' is connected and contains B_ℓ^Q, it follows that $C'' \cap G'' = \emptyset$ and thus $C' = C''$, if G'' and $\bigcup_{i=0}^{3k} B_i^Q$ do not meet.

Indeed, suppose for a contradiction that G'' and $\bigcup_{i=0}^{3k} B_i^Q$ meet in a vertex v. Then

$$d_G(P, q_0) \leqslant d_G(P, v) + d_G(v, q_i) + d_G(q_i, q_0) \leqslant 18k + 4 + 3k + 3k = 24k + 4,$$

which contradicts the choice of q_0.

We can now show that all vertices in C' have distance at most $12k$ from q_0. For this, consider the induced subgraph of G on the vertex set $V(G'') \cup V(C)$. In this graph, q_0 has maximal distance from P and thus we can apply Lemma 4.9 to the shortest $q_0 - P$ path Q, $r = 3k$, $\ell = 3k$ and the component C'. So every vertex in C' has distance at most $9k$ from Q. As every shortest C'–Q path ends in some q_i with $i \leqslant 3k$, this implies that every vertex in C' has distance at most $12k$ from q_0.

Now we obtain a path-decomposition of $G[V(C) \cup V_s]$ as follows. In a first step, for $i \leqslant \ell$ we define

$$U'_i := \bigcup_{q_j \in B_Q(q_i, 3k+1)} B_j^Q.$$

By Lemma 4.2, the U'_i are the bags of a Q-decomposition of G_Q of radial spread at most $4 \cdot (3k) + 2 = 12k + 2$; and every vertex in any U'_i has distance at most $6k + 1$ from q_i in $G[U'_i]$.

In a second step, for $Q^* = Qq_m$, we define a Q^*-decomposition of $G_Q^* \cup C$ where $G_Q^* := G[\bigcup_{i=0}^{m+3k+1} B_i^Q]$. For that, we let $U''_i := U'_i$ for $1 \leqslant i \leqslant m$. The neighbourhood of every component of $C - G_Q$ is contained in U'_0, so we can just add its vertices to U'_0: let U''_0 be the union of U'_0 and all vertices of components of $C - G_Q$. By construction U''_0 has radius at most $12k$. Then the U''_i are the bags of a Q^*-decomposition of $G_Q^* \cup C$. Indeed, every vertex of $C - G_Q$ is contained in U''_0 by definition, and every vertex of $C \cap G_Q$ is contained in G_Q^* since all vertices in $\bigcup_{i=m+3k+2}^{\ell} B_i^Q$ have distance at most $18k + 3$ from P and are thus not contained in C. Moreover, $(Q^*, \mathcal{U}'')$ has radial width at most $12k$ and radial spread at most $12k + 2$.

We now restrict the bags U''_i to C and add V_s to all U''_i with $i \geqslant m - 6k$, i.e. we set $U_i := (U''_i \cap V(C))$ for $i < m - 6k$ and $U_i := (U''_i \cap V(C)) \cup V_s$ for all $i \geqslant m - 6k$. To verify that $(Q^*, \mathcal{U})$ is a decomposition of $G[V(C) \cup V_s]$, it suffices to show that $N_G(G - C) \subseteq U_m$.

For this, let $v \in N_G(G - C) \subseteq V(C)$ be given, and let w be a neighbour of v in $G - C$. As C is a component of $G - G''$, this means that $w \in G''$. As we have already seen above, no component of $C - G_Q$ has neighbours in G'', so $v \in G_Q$, which in particular implies that $v \in G_Q^*$. If $v \in \bigcup_{i=m-3k-1}^{m+3k+1} B_i^Q$, then $v \in U_m$ by construction. So we may assume that $v \in B_j^Q$ for some $j < m - 3k - 1$. But this implies that q_0 has distance at most

$$\begin{aligned} &d_G(q_0, q_j) + d_G(q_j, v) + d_G(v, w) + d_G(w, P) \\ &\qquad \leqslant (m - 3k - 2) + 3k + 1 + (18k + 4) = m + 18k + 3 < \ell \end{aligned}$$

from P, which contradicts that Q is a shortest q_0–P path.

Hence, $(Q^*, \mathcal{U})$ is a path-decomposition of $G[V(C) \cup V_s]$, and it has radial spread at most $12k+2$ by construction. Moreover, it has radial width at most $54k+8$: Let $i \leqslant m$ and let $v \in U_i$. If $i \leqslant m-6k$, then, since $Q^* \subseteq C$ is a shortest q_0–$N_G(G-C)$ path in G and $G[U_i']$ has radius at most $12k$, we have $U_i = U_i'$ and hence $G[U_i]$ has radius at most $12k$. Now suppose that $i > m-6k$. If $v \in V_s$, then v has distance at most $36k+7$ from p_s by Claim 6.6. Otherwise, if $v \in U_i \setminus V_s$ then

$$\begin{aligned} d_{G[U_i]}(v, p_s) &< d_{G[U_i]}(v, q_i) + d_{G[U_i]}(q_i, q_m) + d_{G[U_i]}(q_m, p_s) \\ &\leqslant 12k + 6k + (36k+8) = 54k+8, \end{aligned}$$

where we used that $q_i Q q_m \subseteq G[U_i]$ by definition, that $q_m \in N_G(V_s') \subseteq V_s$ by Claim 6.5, and that every vertex in V_s has distance at most $36k+7$ from p_s by Claim 6.6. ■

We now combine these path-decompositions of the components of $G-V'''$ with the path-decomposition $(P, \mathcal{V})$ of G''' to a star-decomposition of G. For this, recall that by Claim 6.6, $(P, \mathcal{V})$ is a path-decomposition of G''' of radial width at most $36k+7$ and radial spread at most $24k+5$. Moreover, the components in $\mathcal{C}$ are precisely the components of $G-V'''$. For every component $C \in \mathcal{C}$, Claim 6.7 guarantees the existence of a decomposition $(Q_C, \mathcal{U}_C)$ of C modelled on a path Q_C of radial width at most $54k+8$ and radial spread at most $12k+2$ such that the bag in $\mathcal{U}_C$ corresponding to the last vertex of Q_C contains V_s.

We obtain a subdivided star S from the disjoint union of P and the Q_C by adding edges from the last vertices of the Q_C to p_s. Now every vertex h of S already has a bag, which we denote by V_h, in exactly one of the path-decompositions $(P, \mathcal{V})$ or $(Q_C, \mathcal{U}_C)$. It is straightforward to check that $(S, \mathcal{V})$ is a star-decomposition of G and that is radial width is at most $54k+8$. Moreover, its radial spread is at most $(24k+5)+1+6k = 30k+7$, as only vertices in V_s may lie in more than one decomposition of the form $(P, \mathcal{V})$ or $(Q_C, \mathcal{V})$. This completes the proof. □

We remark that the proof actually yields that the subdivided star S is $(1, 60k+14)$-quasi-isometric to G (where we may choose $\varphi(h) = h$ for all vertices h of S as $V(S) \subseteq V(G)$ by construction). Here is a hint for the proof: The proof of Section 6 shows that all vertices in G have distance at most $58k+9 \leqslant 60k+14$ from some vertex of S. Moreover, since P and all paths Q_C are geodesic in G, it remains to check the distances of vertices h, h' of S that lie on distinct such paths. Assume $h \in V(Q_C)$ and $h' \in V(Q_{C'})$ ($h \in V(Q_C)$ and $h' \in V(P)$ is similar). Then $d_S(h, h') \leqslant d_G(h, h')$ as $Q_C \subseteq C$ is a shortest path in G between its first vertex and $N_G(G-C)$.

Also, $d_G(h,h') \leqslant d_S(h,p_s) + d_G(N_G(G-C), N_G(G-C')) + d_S(p_s,h') = d_S(h,h') + 60k + 14$ since p_s is the centre of the subdivided star S and because $d_G(N_G(G-C), N_G(G-C')) \leqslant 2 \cdot (18k+4) + 2 \cdot (12k+3)$ follows from the fact that C, C' are components of $G - V'' = G - B_G(P, 18k+4)$ and from Claim 6.3.

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