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BULL-FREE GRAPHS AND χ-BOUNDEDNESS

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Abstract. — A *bull* is a graph obtained from a four-vertex path by adding a vertex adjacent to the two middle vertices of the path. A graph G is *bull-free* if no induced subgraph of G is a bull. We prove that for all $k, t \in \mathbb{N}$, if G is a bull-free graph of clique number at most k and every triangle-free induced subgraph of G has chromatic number at most t, then G has chromatic number at most $k^{O(\log t)}$. We further show that the bound $k^{O(\log t)}$ is best possible up to a multiplicative constant in the exponent.

Thomassé, Trotignon, and Vušković (2017) were the first to give a bound of the form $2^{p \log p}$, where $p = O(k^2 + t)$, with a proof that uses Chudnovsky's structure theorem for bull-free graphs. This was improved by Chudnovsky, Cook, Davies, and Oum (2026) to a bound of the form $k^{O(t)}$, with a 10-page proof that again relies heavily on Chudnovsky's structure theorem.

Our proof is a single page long and completely avoids the structure theorem, instead using only a result of Chudnovsky and Safra (which itself has a short proof).

1. Introduction

We denote the set of all positive integers by $\mathbb{N}$. Graphs in this paper have finite vertex sets, no loops, and no parallel edges. Given a graph $G = (V(G), E(G))$, the chromatic number and the clique number of G are denoted, respectively, by $\chi(G)$ and $\omega(G)$. A graph class $\mathcal{C}$ is *hereditary* if $\mathcal{C}$ is closed under taking induced subgraphs, and $\mathcal{C}$ is (*polynomially*) *χ-bounded* if there is a (polynomial) function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ for every $G \in \mathcal{C}$. The study of χ-bounded classes is a central topic in structural graph theory; see [15, 16] for surveys. This work focuses on χ-boundedness in the class of bull-free graphs. A *bull* is a graph obtained from a four-vertex path by adding a vertex adjacent to the two middle vertices of the path. A graph G is *bull-free* if no induced subgraph of G is a bull.

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The class of bull-free graphs is not χ-bounded since triangle-free graphs are bull-free, and there are triangle-free graphs of arbitrarily large chromatic number [13]. Thomassé, Trotignon, and Vušković [17] proved that the latter is in fact the only obstruction to χ-boundedness for bull-free graphs. For all $t \in \mathbb{N}$, let $\mathcal{B}_t$ be the class of all bull-free graphs in which every triangle-free induced subgraph has chromatic number at most t.

Theorem 1.1 (Thomassé, Trotignon, Vušković; [17, 8.1 and 8.2]). — *For all $t \in \mathbb{N}$, every graph $G \in \mathcal{B}_t$ satisfies $\chi(G) \leqslant p^p$ with $p = O(\omega(G)^2 + t)$.*

One motivation for Theorem 1.1 was a conjecture [14] that *every* hereditary class in which triangle-free graphs have bounded chromatic number is χ-bounded. This was later refuted by Carbonero, Hompe, Moore, and Spirkl [3]. Soon after, Briański, Davies, and Walczak [2] adapted the method of [3] and disproved another conjecture, due to Esperet [11], that every χ-bounded hereditary class is polynomially χ-bounded. Recently, Chudnovsky, Cook, Davies, and Oum [8] proved that Esperet's conjecture holds in the class of bull-free graphs (they also proved the same for several other hereditary classes, yet declared bull-free graphs as "the most difficult case" [8]). In fact, they proved the following:

Theorem 1.2 (Chudnovsky, Cook, Davies, Oum; [8, 1.2(v)]). — *For all $t \in \mathbb{N}$, the class $\mathcal{B}_t$ is polynomially χ-bounded.*

The proofs of both Theorem 1.1 and Theorem 1.2 rely on the structure theorem for bull-free graphs. This is a difficult result due to Chudnovsky, whose statement alone requires several pages of definitions, and whose proof is spread across a series of four papers [4, 5, 6, 7]. The proof of Theorem 1.2, in particular, takes over 10 pages and uses several results from [4, 5, 6, 7] on the structure of bull-free graphs and "trigraphs." As for the bound, it is mentioned in [8] that a careful analysis of their proof gives $\chi(G) \leqslant \omega(G)^{O(t)}$ for all $G \in \mathcal{B}_t$.

Our main result is the following strengthening of Theorem 1.2:

Theorem 1.3. — *We have $\chi(G) \leqslant \omega(G)^{4\log t+13}$ for all $t \in \mathbb{N}$ and every graph $G \in \mathcal{B}_t$.*

This has two advantages. First, the proof is a single page long. It entirely bypasses the material from [4, 5, 6, 7] and combines only one result of Chudnovsky and Safra [10] (which has a proof no longer than five pages) with a "layering argument." We remark that it is quite unusual for a layering proof to yield polynomial bounds. Second, not only does Theorem 1.3 give an exponential improvement in the degree of the polynomial bound claimed in [8], it turns out that logarithmic degree in t is asymptotically best possible:

THEOREM 1.4. — *For all $t \in \mathbb{N}$, every polynomial χ-bounding function for $\mathcal{B}_t$ has degree larger than $\frac{1}{2}\log t$.*

We will prove Theorem 1.3 and Theorem 1.4 in Sections 2 and 3, respectively.

2. The upper bound

Let G be a graph. For $X \subseteq V(G)$, we use X to denote both the set X and the induced subgraph of G with vertex set X. For $v \in V(G)$, we write $N_X(v)$ for the set of all neighbors of v in X. We say that X is a *homogeneous set in* G if $1 < |X| < |V(G)|$ and for every $v \in V(G) \setminus X$, either $N_X(v) = X$ or $N_X(v) = \varnothing$. We say that G is *prime* if there is no homogeneous set in G. Recall that a graph G is *perfect* if $\chi(H) \leqslant \omega(H)$ for every induced subgraph H of G. We say that G is *N-perfect* if for every $v \in V(G)$, either $N_G(v)$ or $G \setminus N_G(v)$ is perfect. Observe that every induced subgraph of an N-perfect graph is also N-perfect.

We only use one result on the structure of bull-free graphs, which has a fairly short proof:

THEOREM 2.1 (Chudnovsky and Safra; [10, 1.4 and 4.3]). — *Every prime bull-free graph is N-perfect.*

We also need the following. A similar result from [9] was also used in [8] to prove Theorem 1.2.

THEOREM 2.2 (Bourneuf and Thomassé; [1, 6.1]). — *Let $\mathcal{C}$ be a hereditary class and let $k \in \mathbb{N}$ such that $\chi(G) \leqslant \omega(G)^k$ for every prime graph $G \in \mathcal{C}$. Then $\chi(G) \leqslant \omega(G)^{2k+3}$ for every graph $G \in \mathcal{C}$.*

We are now ready to prove our main result:

Proof of Theorem 1.3. — We may assume that $t \geqslant 2$ (because $\chi(G) = \omega(G) = 1$ for all $G \in \mathcal{B}_1$). By Theorem 2.1 and Theorem 2.2, it suffices to show that $\chi(G) \leqslant \omega(G)^{2\log t+5}$ for every N-perfect graph $G \in \mathcal{B}_t$. The proof is by induction on $\omega(G)$, and we may assume that $\omega(G) \geqslant 2$ and G is connected. Suppose that $G \setminus N_G(u)$ is perfect for some $u \in V(G)$. Since $\omega(N_G(u)) \leqslant \omega(G) - 1$, it follows from the inductive hypothesis that $\chi(N_G(u)) \leqslant (\omega(G)-1)^{2\log t+5}$. But now $\chi(G) \leqslant (\omega(G)-1)^{2\log t+5} + \omega(G) \leqslant \omega(G)^{2\log t+5}$, as desired.

Henceforth, assume that $N_G(u)$ is perfect for all $u \in V(G)$. Let $v \in V(G)$ be fixed. For each $r \in \mathbb{N} \cup \{0\}$, let L_r be the set of all vertices at distance exactly r from v in G. Note that $L_0 = \{v\}$ and $L_1 = N_G(v)$ (and $L_r = \varnothing$ for all sufficiently large r).

(1) *Let $r \in \mathbb{N}$, let H be a prime induced subgraph of L_r, and let $x \in L_{r-1}$. Then either $N_H(x) = H$ or $N_H(x)$ is a stable set.*

If $r = 1$, then $x = v$ and $N_H(x) = N_H(v) = H$. Assume that $r \geqslant 2$. Then x has a neighbor $y \in L_{r-2}$. Suppose for a contradiction that $|N_H(x)| < |H|$ and $N_H(x)$ is not a stable set. Then $N_H(x)$ has a component K with $1 < |K| \leqslant |N_H(x)| < |H|$. Since K is not a homogeneous set in H, there is a vertex in $H \setminus K$ with both a neighbor and a non-neighbor in K. Since K is a component of $N_H(x)$, there is a vertex $w \in H \setminus N_H(x)$ as well as distinct and adjacent vertices $u, u' \in K$ such that $uw \in E(G)$ and $u'w \notin E(G)$. But now $\{u, u', w, x, y\}$ induces a bull in G, a contradiction. This proves (1).

(2) *For every $r \in \mathbb{N} \cup \{0\}$, we have $\chi(L_r) \leqslant \omega(G)^{2\log t+4}$.*

This is immediate for $r \in \{0, 1\}$ because L_r is perfect. Assume that $r \geqslant 2$. For every $a \in \{1, \dots, r\}$, we define $S_1^a, \dots, S_{\omega(G)}^a \subseteq L_a$ recursively, as follows. For $a = 1$, let $S_1^1, \dots, S_{\omega(G)}^1 \subseteq L_1$ be (possibly empty) stable sets with union L_1, which exist since $L_1 = N_G(v)$ is perfect. For $a \in \{2, \dots, r\}$, having defined $S_1^{a-1}, \dots, S_{\omega(G)}^{a-1} \subseteq L_{a-1}$, let S_b^a, for each $b \in \{1, \dots, \omega(G)\}$, be the set of all vertices in L_a with a neighbor in S_b^{a-1}. Note that for every $a \in \{1, \dots, r\}$, the sets $S_1^a, \dots, S_{\omega(G)}^a \subseteq L_a$ have union L_a. So in order to prove (2), it is enough to show that $\chi(S_b^r) \leqslant \omega(G)^{2\log t+3}$ for every $b \in \{1, \dots, \omega(G)\}$. To that end, by Theorem 2.2, it suffices to show that every prime induced subgraph H of $S_b^r \subseteq L_r$ satisfies $\chi(H) \leqslant \omega(H)^{\log t}$. If $\omega(H) \leqslant 1$, then $\chi(H) \leqslant 1$. So assume that $\omega(H) \geqslant 2$. Also, if there is a vertex $x \in L_{r-1}$ for which $N_H(x) = H$, then H is perfect and we are done (because $t \geqslant 2$). Thus, by (1), we may assume that $N_H(x)$ is stable for every $x \in L_{r-1}$. We claim that $\omega(H) = 2$. Suppose not. Then there is a triangle $\{u_1, u_2, u_3\}$ in H. For each $i \in \{1, 2, 3\}$, choose a neighbor $x_i \in S_b^{r-1} \subseteq L_{r-1}$ of u_i; note that u_i is the only neighbor of x_i in $\{u_1, u_2, u_3\}$ because $N_H(x_i)$ is stable. In particular, x_1, x_2, x_3 are pairwise distinct. If there are distinct $i, j \in \{1, 2, 3\}$ for which $x_i x_j \notin E(G)$, then $\{u_1, u_2, u_3, x_i, x_j\}$ induces a bull in G, a contradiction. So $\{x_1, x_2, x_3\} \subseteq S_b^{r-1} \subseteq L_{r-1}$ is a triangle in G. Since S_b^1 is stable, it follows that $r \geqslant 3$, which in turn implies that x_1 has a neighbor $y \in L_{r-2}$, and y has a neighbor $z \in L_{r-3}$. Now, if $x_2 y, x_3 y \notin E(G)$, then $\{u_2, x_1, x_2, x_3, y\}$ induces a bull in G, and if $x_i y \in E(G)$ for some $i \in \{2, 3\}$, then $\{u_1, x_1, x_i, y, z\}$ induces a bull in

G, a contradiction. The claim that $\omega(H) = 2$ follows. Since $G \in \mathcal{B}_t$ and $\omega(H) = 2$, it follows that $\chi(H) \leqslant t = \omega(H)^{\log t}$. This proves (2).

Note that $\chi(G) \leqslant 2 \max_{r \geqslant 0} \chi(L_r)$ because G is connected. Hence, by (2) and since $\omega(G) \geqslant 2$, we have $\chi(G) \leqslant \omega(G)^{2 \log t + 5}$. □

3. The lower bound

The proof of Theorem 1.4 uses the "Mycielski construction" [13], which yields a sequence $\{M_n\}_{n \geqslant 0}$ of graphs such that for each $n \in \mathbb{N}$, we have $\omega(M_n) = 2$ and $\chi(M_n) = n + 1$ (the construction is well-known and we do not need the definition). There is also a beautiful recursion for the "fractional chromatic number" of the Mycielski graphs [12] (again, we do not need the definition of the fractional chromatic number). For each $n \in \mathbb{N}$, let ϕ_n be the fractional chromatic number of M_n.

Theorem 3.1 (Larsen, Propp, Ullman [12]). — *We have $\phi_1 = 2$ and $\phi_{n+1} = \phi_n + \phi_n^{-1}$ for every $n \in \mathbb{N}$.*

Corollary 3.2. — *For every $n \in \mathbb{N}$, we have $\phi_n \geqslant \sqrt{2(n+1)}$.*

Proof. — This is clear for $n = 1$. Assume that $n \geqslant 2$. Then by Theorem 3.1, we have

$$\begin{aligned}\phi_n^2 = \phi_1^2 + \sum_{k=1}^{n-1} \left(\phi_{k+1}^2 - \phi_k^2\right) &= 4 + \sum_{k=1}^{n-1} \left(2 + \phi_k^{-2}\right) \\ &\geqslant 4 + 2(n-1) = 2(n+1).\end{aligned}$$ □

Let G_1 and G_2 be graphs with disjoint non-empty vertex sets, and let $x \in V(G_1)$. A graph G is obtained by *substituting G_2 for x in G_1* if $V(G) = (V(G_1) \setminus \{x\}) \cup V(G_2)$, and the following hold:

- For all $u, v \in V(G_1) \setminus \{x\}$, we have $uv \in E(G)$ if and only if $uv \in E(G_1)$.
- For all $u, v \in V(G_2)$, we have $uv \in E(G)$ if and only if $uv \in E(G_2)$.
- Each vertex $v \in N_{G_1}(x)$ is adjacent in G to all vertices in $V(G_2)$.
- Each vertex $v \in V(G_1) \setminus (N_{G_1}(x) \cup \{x\})$ is non-adjacent in G to all vertices in $V(G_2)$.

In particular, if $|V(G_1)| = 1$, then $G = G_2$, if $|V(G_2)| = 1$, then $G = G_1$, and if $|V(G_1)|, |V(G_2)| > 1$, then $V(G_2)$ is a homogeneous set in G. Given a graph class $\mathcal{C}$, we write $\mathcal{C}^*$ for the closure of $\mathcal{C}$ under disjoint union and substitution. Observe that if $\mathcal{C}$ is hereditary, then so is $\mathcal{C}^*$.

Theorem 3.3 (Chudnovsky, Penev, Scott, Trotignon; [9, 2.4]). — *Let $d \in \mathbb{N}$ and let H be a triangle-free graph of fractional chromatic number larger than 2^d. Let $\mathcal{C}$ be the union of the class of all induced subgraphs of H and the class of all complete graphs. Then every polynomial χ-bounding function for $\mathcal{C}^*$ has degree larger than d.*

We now prove Theorem 1.4:

Proof of Theorem 1.4. — The result is immediate for $t = 1$. Assume that $t \geqslant 2$. Let f be a polynomial χ-bounding function for $\mathcal{B}_t$ of degree $\delta \in \mathbb{N}$. Let $\mathcal{I}$ be the class of all induced subgraphs of M_{t-1} and let $\mathcal{C}$ be the union of $\mathcal{I}$ and the class of all complete graphs.

Observe that $\mathcal{C}$ is hereditary, and so $\mathcal{C}^*$ is hereditary. Also, every graph in $\mathcal{C}$ is bull-free. Since a bull is connected and has no homogeneous set, it follows that every graph in $\mathcal{C}^*$ is bull-free. Moreover, since $\mathcal{I}$ is exactly the class of all triangle-free graphs in $\mathcal{C}$, it follows that the class of all triangle-free graphs in $\mathcal{C}^*$ is exactly the closure of $\mathcal{I}$ under disjoint union and substitution of stable sets for vertices. Since $\chi(M_{t-1}) = t \geqslant 2$, it follows that $\chi(G) \leqslant t$ for every triangle-free graph $G \in \mathcal{C}^*$. We deduce that $\mathcal{C}^* \subseteq \mathcal{B}_t$, and so f is a polynomial χ-bounding function for $\mathcal{C}^*$.

On the other hand, by Corollary 3.2, we have $\phi_{t-1} \geqslant \sqrt{2t} = 2^{\frac{1}{2}(\log t+1)} > 2^{\lfloor \frac{1}{2}\log t \rfloor}$. Hence, by Theorem 3.3, we have $\delta > \lfloor \frac{1}{2} \log t \rfloor$, and thus $\delta > \frac{1}{2} \log t$ since δ is an integer. $\square$

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