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INNOVATIONS IN GRAPH THEORY

A NOTE ON MATCHING VARIABLES TO EQUATIONS

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Abstract. — We showed with J. P. Gollin that if a (possibly infinite) homogeneous linear equation system has only the trivial solution, then there exists an injective function from the variables to the equations such that each variable appears with non-zero coefficient in its image. Shortly after, a more elementary proof was found by Aharoni and Guo. In this note we present a very short matroid-theoretic proof of this theorem.

1. Introduction

Let $\mathbb{F}$ be a field, and $A \in \mathbb{F}^{m\times n}$. Suppose that the homogenous linear equation system $Ax = 0$ has only the trivial solution $x = 0$. Then for each subset C of the columns of A, there must be at least $|C|$ rows where at least one column in C have a non-zero element. Indeed, otherwise, the columns in C are linearly dependent which leads to a non-trivial solution of $Ax = 0$. It follows by Hall's theorem that the columns can be matched to the rows along non-zero elements. More precisely, there is an injection $\varphi : [n] \to [m]$ such that $a_{\varphi(j),j} \neq 0$ for every $j \in [n]$.

We investigated with J. P. Gollin if this remains true for infinite equation systems (with finitely many variables in each equation). Although Hall's marriage theorem has a certain generalization for infinite graphs (see [2, Theorem 3.2]), the argument above does not seem to be adaptable to the infinite case. Even so, other tools in infinite matching theory (namely [10, Theorem 1]) let us answer the question affirmatively:

Theorem 1.1. — *Let $\mathbb{F}$ be a field, and let $a : I \times J \to \mathbb{F}$ be a function such that for each fixed $i \in I$, there are only finitely many $j \in J$ such that*

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$a_{i,j} \neq 0$. Suppose that the homogenous linear equation system defined by a has only the trivial solution, i.e. $\sum_{j\in J} a_{i,j}x_j = 0$ for each $i \in I$ for a function $x : J \to \mathbb{F}$ only if x is constant zero. Then there is an injection $\varphi : J \to I$ such that $a_{\varphi(j),j} \neq 0$ for every $j \in J$.

Shortly after, a more elementary proof was found by Aharoni and Guo [3]. They also pointed out a short matroid-theoretic proof in the finite case. This made us wonder if a similar matroid-theoretic approach could be successful for infinite equation systems. The key tools we use in this note are the so called thin-sum matroids introduced by Bruhn and Diestel (see [6, Theorem 18]) and further investigated by Afzali and Bowler in [1]. These together with (the dualization of) a base exchange property of infinite matroids due to Aharoni and Pouzet [4, Theorem 2.1] leads to a short proof of Theorem 1.1. The aim of this note is to present this proof.

2. Preliminaries

Infinite matroids were introduced by Higgs in the late 1960s [8]. Several decades later the same concept was discovered independently by Bruhn et al. in [7] together with the following axiomatization: A *matroid* is an ordered pair $M = (E, \mathcal{I})$ with $\mathcal{I} \subseteq \mathcal{P}(E)$ such that

(I) $\emptyset \in \mathcal{I}$;
(II) $\mathcal{I}$ is closed under taking subsets;
(III) For every $I, J \in \mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ and I is not, there exists an $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$;
(IV) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

The sets in $\mathcal{I}$ are called *independent* while the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are *dependent*. The maximal independent sets (exists by (IV)) are called *bases*. The minimal dependent sets are the *circuits*. Every dependent set contains a circuit (which is a non-trivial fact for infinite matroids). For an $X \subseteq E$, the pair $M \upharpoonright X := (X, \mathcal{I} \cap \mathcal{P}(X))$ is a matroid and it is called the *restriction* of M to X. We write $M - X$ for $M \upharpoonright (E \setminus X)$ and call it the minor obtained by the *deletion* of X. The *contraction* of X in M is the matroid M/X on $E \setminus X$ in which $I \subseteq E \setminus X$ is independent iff $J \cup I$ is independent in M for a (equivalently: for every) set J that is maximal among the independent subset of X. Contraction and deletion commute, i.e. for disjoint $X, Y \subseteq E$, we have $(M/X) - Y = (M - Y)/X$. Matroids of this form are the *minors* of M. We say that $X \subseteq E$ *spans* $e \in E$ in matroid M if either $e \in X$ or

$\{e\}$ is dependent in M/X. The dual M^* of M is the matroid on the same ground set in which a set is independent if it is disjoint from a base of M. Contraction and deletion are related by duality in the following way: $(M/X-Y)^* = M^*/Y-X$. A matroid is called *finitary* if all its circuits are finite and it is *cofinitary* if its dual is finitary. The class of finitary matroids is closed under taking minors and thus so is the class of cofinitary matroids.

THEOREM 2.1 (Aharoni and Pouzet, [4, Theorem 2.1]). — *If M is a finitary matroid, then for every bases B_0 and B_1 of M, there is a bijection $f : B_0 \to B_1$ such that $B_0 \setminus \{x\} \cup \{f(x)\}$ is a base of M for each $x \in B_0$.*

For short proofs of generalizations of Theorem 2.1 using the same notation we are using in this article, see [9].

The following concept of thin-sum matroids was introduced by Bruhn and Diestel (see [6, Theorem 18]): Let X be a set and let $\mathbb{F}$ be a field. A family $\{f_e : e \in E\}$ of $X \to \mathbb{F}$ functions is called *thin* if for each $x \in X$ there are only finitely many $e \in E$ with $f_e(x) \neq 0$. Note that for any $\lambda : E \to \mathbb{F}$, the function $\sum_{e\in E} \lambda_e f_e$ is well-defined pointwise. An $I \subseteq E$ is *thin independent* when $\sum_{e\in I} \lambda_e f_e$ is the constant zero function only if $\lambda_e = 0$ for each $e \in I$.

THEOREM 2.2 (Afzali and Bowler [1, Corollary 3.4]). — *The notion of thin independence in a thin family of functions gives rise to a cofinitary matroid.*

For more information about infinite matroids, we refer to [5].

3. Proof of the main result

We show by a relatively simple dualization argument that Theorem 2.1 remains true for cofinitary matroids.

THEOREM 3.1. — *If M is a cofinitary matroid, then for every bases B_0 and B_1 of M, there is a bijection $f : B_0 \to B_1$ such that $B_0 \setminus \{x\} \cup \{f(x)\}$ is a base of M for each $x \in B_0$.*

Proof. — First we show that we can assume without loss of generality that B_0 and B_1 are disjoint and $B_0 \cup B_1 = E(M)$. Indeed, suppose we already know this special case of the theorem and let M' be the matroid that we obtain from M by contracting $B_0 \cap B_1$ and deleting $E \setminus (B_0 \cup B_1)$. Then M' is still cofinitary, moreover, $B_0 \setminus B_1$ and $B_1 \setminus B_0$ are disjoint bases of M' whose union is $E(M')$. Let f' be a function that we obtain by applying the assumed special case of the theorem with M', $B_0 \setminus B_1$ and $B_1 \setminus B_0$. Then the extension f of f' to B_0 where $f(x) := x$ for every $x \in B_0 \cap B_1$ is as desired.

Note that M^* is a finitary matroid and under our assumption B_0 and B_1 are bases of M^*. Let $g : B_1 \to B_0$ be a bijection that we obtain by applying Theorem 2.1 with M^*, B_1 and B_0 (i.e. the roles of the bases are switched). Then $B_1 \setminus \{y\} \cup \{g(y)\}$ is a base of M^* for each $y \in B_1$. This means that its complement $B_0 \setminus \{g(y)\} \cup \{y\}$ is a base of M for each $y \in B_1$. By substituting y with $g^{-1}(x)$, we conclude that $B_0 \setminus \{x\} \cup \{g^{-1}(x)\}$ is a base of M for each $x \in B_0$. Thus $f := g^{-1}$ is suitable. □

Lemma 3.2. — *For a matroid M the following are equivalent:*

(i) For every bases B_0 and B_1 of M, there is a bijection $f : B_0 \to B_1$ such that $B_0 \setminus \{x\} \cup \{f(x)\}$ is a base of M for each $x \in B_0$.
(ii) For every bases B_0 and B_1 of M, there is a bijection $f : B_0 \to B_1$ such that $B_1 \setminus \{f(x)\} \cup \{x\}$ is a base of M for each $x \in B_0$.

Proof. — To derive one property from the other, apply the assumed property while switching the roles of B_0 and B_1, then take the inverse of the resulting function. □

Corollary 3.3. — *If M is a cofinitary matroid, J is independent in M and B is a base of M, then there is an injection $f : J \to B$ such that $B \setminus \{f(x)\} \cup \{x\}$ is a base for each $x \in B$.*

Proof. — Extend J to a base and apply Theorem 3.1 combined with Lemma 3.2. □

Proof of Theorem 1.1. — For $j \in J$, let $f_j : I \to \mathbb{F}$ be defined as $f_j(i) := a_{i,j}$. For $i \in I$, let $f_i : I \to \mathbb{F}$ be the function for which $f_i(i) = 1$ and $f_i(i') = 0$ for $i' \neq i$.[(1)] Then $\{f_k : k \in I \cup J\}$ is a thin family of functions, thus by Theorem 2.2 it defines a cofinitary matroid M on $I \cup J$ via thin independence. Moreover, I is base of M and J is independent in it. Thus Corollary 3.3 applied with J and base I gives an injection $\varphi : J \to I$ such that $I \setminus \{\varphi(j)\} \cup \{j\}$ is a base of M for each $j \in J$. But then we must have $f_j(\varphi(j)) \neq 0$ since otherwise $\varphi(j)$ is not spanned by $I \setminus \{\varphi(j)\} \cup \{j\}$ in M because $f_{\varphi(j)}(\varphi(j)) = 1$ while $f_k(\varphi(j)) = 0$ for every $k \in (I \setminus \{\varphi(j)\} \cup \{j\})$, contradicting that $I \setminus \{\varphi(j)\} \cup \{j\}$ is a base of M. By definition this means that $a_{\varphi(j),j} \neq 0$ for every $j \in J$ as desired. □

(1) We assume that the index sets I and J are disjoint.

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