Innov. Graph Theory **1**, 2024, pp. 39–86
https://doi.org/10.5802/igt.4

POWERS OF PLANAR GRAPHS, PRODUCT STRUCTURE, AND BLOCKING PARTITIONS

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Abstract. — We prove that the k-power of any planar graph G is contained in $H \boxtimes P \boxtimes K_{f(\Delta(G),k)}$ for some graph H with bounded treewidth, some path P, and some function f. This resolves an open problem of Ossona de Mendez. In fact, we prove a more general result in terms of shallow minors that implies similar results for many 'beyond planar' graph classes, without dependence on $\Delta(G)$. For example, we prove that every k-planar graph is contained in $H \boxtimes P \boxtimes K_{f(k)}$ for some graph H with bounded treewidth and some path P, and some function f. This resolves an open problem of Dujmović, Morin and Wood. We generalise all these results for graphs of bounded Euler genus, still with an absolute bound on the treewidth.

At the heart of our proof is the following new concept of independent interest. An *ℓ-blocking partition* of a graph G is a partition of $V(G)$ into connected sets such that every path of length greater than ℓ in G contains at least two vertices in one part. We prove that for some constant $\ell \geqslant 1$ every graph of Euler genus g has an ℓ-blocking partition with parts of size bounded by a function of $\Delta(G)$ and g. Motivated by this result, we study blocking partitions in their own right. We show that every graph G has a 2-blocking partition with parts of size bounded by a function of $\Delta(G)$ and $\operatorname{tw}(G)$. On the other hand, we show that 4-regular graphs do not have ℓ-blocking partitions with bounded size parts.

1. Introduction

Graph product structure theory describes complicated graphs as subgraphs of strong products[(1)] of simpler building blocks, which typically have bounded treewidth[(2)]. For example, Dujmović, Joret, Micek, Morin,

Keywords: graph, planar graph, product structure, power, blocking partition, surface, minor.
2020 *Mathematics Subject Classification:* 05C10.

(1) The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

(2) Let $\operatorname{tw}(H)$ denote the treewidth of a graph H (defined in Section 2).

Ueckerdt and Wood [16] proved the following product structure theorem for planar graphs, where a graph H is *contained* in a graph G if H is isomorphic to a subgraph of G.

Theorem 1.1 ([16]). — *Every planar graph is contained in $H \boxtimes P \boxtimes K_3$ for some planar graph H with $\operatorname{tw}(H) \leqslant 3$ and for some path P.*

This result has been the key to solving several long-standing open problems about queue layouts [16], nonrepetitive colourings [13], centred colourings [5], adjacency labelling [21, 12], twin-width [2, 30, 29], vertex ranking [3], and box dimension [19]. Theorem 1.1 has been extended in various ways for graphs of bounded Euler genus [16, 9, 30], graphs excluding an apex minor [16, 28, 14], graphs excluding an arbitrary minor [16, 28, 4], graphs of bounded tree-width [4, 14], graphs of bounded path-width [15], and for various non-minor-closed classes [18, 26].

Many of these results show that for a particular graph class $\mathcal{G}$ there are integers t, c such that every graph in $\mathcal{G}$ is contained in $H \boxtimes P \boxtimes K_c$ for some graph H with treewidth t and for some path P. Here the primary goal is to minimise t, where minimising c is a secondary goal. This paper proves product structure theorems of this form for powers of planar graphs and for various beyond planar graph classes. The distinguishing feature of our results is that $\operatorname{tw}(H)$ is bounded by an absolute constant, instead of depending on a parameter defining $\mathcal{G}$. This is important because in several applications of such product structure theorems, the main dependency is on $\operatorname{tw}(H)$; see Section 1.3 for an example.

First consider powers of planar graphs. For $k \in \mathbb{N}$, the *k-power* of a graph G, denoted G^k, is the graph with vertex-set $V(G)$, where $vw \in E(G^k)$ if and only if $\operatorname{dist}_G(v, w) \in \{1, \dots, k\}$. Dujmović, Morin and Wood [18] proved that for every planar graph G of maximum degree Δ, the k-power G^k is contained in $H \boxtimes P \boxtimes K_{6\Delta^k(k^4+3k^2)}$ for some graph H with $\operatorname{tw}(H) \leqslant \binom{k+3}{3} - 1$ and some path P. Dependence on Δ is unavoidable since, for example, if G is the complete $(\Delta-1)$-ary tree of height k, then G^{2k} is a complete graph on roughly $(\Delta-1)^k$ vertices. Ossona de Mendez [32] asked whether this bound on $\operatorname{tw}(H)$ could be made independent of k. In particular:

Question 1.2 ([32]). — *Is there a constant t and a function f such that for every planar graph G and $k \in \mathbb{N}$, the k-power G^k is contained in $H \boxtimes P \boxtimes K_{f(k,\Delta(G))}$ for some graph H with $\operatorname{tw}(H) \leqslant t$ and for some path P?*

We resolve this question, in the following strong sense. For integers $k, d \geqslant 1$ and a graph G, let G^k_d be the graph with vertex-set $V(G)$ where $vw \in$

$E(G_d^k)$ whenever there is a vw-path P in G of length at most k such that every internal vertex of P has degree at most d in G. The following theorem answers Question 1.2 in the affirmative, since $G^k = G^k_{\Delta(G)}$.

Theorem 1.3. — *There is a function f such that for every planar graph G and for any integers $k, d \geqslant 1$, the graph G_d^k is contained in $H \boxtimes P \boxtimes K_{f(k,d)}$ for some graph H with $\operatorname{tw}(H) \leqslant 15\,288\,899$ and for some path P.*

We chose to simplify the proof instead of optimising the constant upper bound on $\operatorname{tw}(H)$ in Theorem 1.3 and in our other results.

Theorem 1.3 is in fact a corollary of a more general result expressed in terms of shallow minors.

1.1. Shallow Minors and Beyond Planar Graphs

Let G and H be graphs and let $r, s \geqslant 0$ be integers. H is a *minor* of G if a graph isomorphic to H can be obtained from G by vertex deletion, edge deletion, and edge contraction. A class $\mathcal{G}$ of graphs is *minor-closed* if for every $G \in \mathcal{G}$ every minor of G is in $\mathcal{G}$. A *model* $(B_x : x \in V(H))$ of H in G is a collection of vertex-disjoint connected subgraphs in G such that B_x and B_y are adjacent in G for every edge $xy \in E(H)$. Clearly H is a minor of G if and only if G contains a model of H. If there exists a model of H in G such that B_x has radius at most r for all $x \in V(H)$, then H is an *r-shallow minor* of G. A *rooted model* $((B_x, v_x) : x \in V(H))$ of H is a model of H where each B_x has a corresponding root $v_x \in V(B_x)$. If for every $x \in V(H)$ and for every $u \in V(B_x) \setminus \{v_x\}$, we have $\operatorname{dist}_{B_x}(v_x, u) \leqslant r$ and $\deg_{B_x}(u) \leqslant s$, then $((B_x, v_x) : x \in V(H))$ is an *(r,s)-shallow model* and H is an *(r,s)-shallow minor* of G. Clearly, if H is an r-shallow minor of G, then H is an $(r, \Delta(G))$-shallow minor of G. However, these definitions do not assume $\Delta(G)$ is bounded, since each vertex v_x may have unbounded degree in B_x and each vertex $u \in V(B_x)$ may have unbounded degree in G.

Building on the work of Dujmović *et al.* [18], Hickingbotham and Wood [26] showed that shallow minors inherit product structure.

Theorem 1.4 ([26]). — *If a graph G is an r-shallow minor of $H \boxtimes P \boxtimes K_c$ where $\operatorname{tw}(H) \leqslant t$, then G is contained in $J \boxtimes P \boxtimes K_{c(2r+1)^2}$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{2r+1+t}{t} - 1$.*

Our main contribution is the following product structure theorem for (r,s)-shallow minors of planar graphs, where J has treewidth bounded by an absolute constant.

Theorem 1.5. — *There is a function f such that for every planar graph G and for every (r,s)-shallow minor H of $G \boxtimes K_d$, H is contained in $J \boxtimes P \boxtimes K_{f(d,r,s)}$ for some graph J with $\operatorname{tw}(J) \leqslant 15\,288\,899$ and for some path P.*

Theorem 1.5 is useful since, as observed by Hickingbotham and Wood [26], many non-minor-closed graph classes can be described as shallow minors of a strong product of a planar graph with a small complete graph. For example, for any graph G with maximum degree Δ, Hickingbotham and Wood [26] observed that G^k is a $\lfloor \frac{k}{2} \rfloor$-shallow minor of $G \boxtimes K_{\Delta^{\lfloor k/2 \rfloor +1}}$. The proof is readily adapted[(3)] to show that G^k_d is a $(\lfloor \frac{k}{2} \rfloor, d)$-shallow minor of $G \boxtimes K_{d^{\lfloor k/2 \rfloor +1}}$. Thus Theorem 1.5 implies Theorem 1.3.

Theorem 1.5 can also be applied to several well-studied beyond planar graph classes, which we now introduce. See [6, 27] for surveys on beyond planarity.

A graph G is *k-planar* if G has a drawing in the plane in which each edge is involved in at most k crossings, where no three edges cross at a single point; such graphs are widely studied, see [35, 18, 11, 17] for example. Dujmović *et al.* [18] proved that every k-planar graph is contained in $H \boxtimes P \boxtimes K_{18k^2+48k+30}$ for some graph H of treewidth $\binom{k+4}{3} - 1$ and for some path P. Dujmović *et al.* [18] asked whether this bound on $\operatorname{tw}(H)$ could be made independent of k. In particular:

Question 1.6 ([18]). — *Is there a constant t and a function f such that every k-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ for some graph H with $\operatorname{tw}(H) \leqslant t$?*

Theorem 1.5 resolves this question.

Corollary 1.7. — *There is a function f such that every k-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 15\,288\,899$.*

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(3) Let D be the set of vertices with degree at most d in G. For each vertex $v \in V(G)$, let B'_v be the subgraph of G induced by the set of vertices $x \in V(G)$ for which there is a vx-path P in G of length at most $\lfloor \frac{k}{2} \rfloor$ such that $V(P - v) \subseteq D$. So the radius of B'_v is at most $\lfloor \frac{k}{2} \rfloor$ and there is an edge between $V(B'_u)$ and $V(B'_v)$ in G for each $uv \in E(G^k)$. Furthermore, $|\{v \in V(G) \colon x \in V(B'_v)\}| \leqslant d^0 + \cdots + d^{\lfloor k/2 \rfloor} \leqslant d^{\lfloor k/2 \rfloor +1}$ for each $x \in V(G)$. So an arbitrary injective map from $\{v \in V(G) \colon x \in V(B'_v)\}$ to $V(K_{d^{\lfloor k/2 \rfloor +1}})$ for each vertex $x \in V(G)$ defines a subgraph B_v of $G \boxtimes K_{d^{\lfloor k/2 \rfloor +1}}$ such that the projection of B_v onto G is B'_v and $V(B_v) \cap V(B_u) = \varnothing$ for all distinct $u, v \in V(G)$. So $(B_v \colon v \in V(G^k_d))$ defines a model of G^k_d in $G \boxtimes K_{d^{\lfloor k/2 \rfloor +1}}$ where each B_v has radius at most $\lfloor \frac{k}{2} \rfloor$, as required. By construction, G^k_d is in fact a $(\lfloor \frac{k}{2} \rfloor, d)$-shallow minor of $G \boxtimes K_{d^{\lfloor k/2 \rfloor +1}}$.

Proof. — Hickingbotham and Wood [26] observed that G is a $\lceil\frac{k}{2}\rceil$-shallow minor of $H \boxtimes K_2$, where H is the planar graph obtained from G by adding a dummy vertex at each crossing point. A close inspection of their proof reveals that each branch set in the model of G in $H \boxtimes K_2$ is a subdivided star rooted at the high degree vertex. So G is a $(\lceil\frac{k}{2}\rceil, 2)$-shallow minor of $H \boxtimes K_2$. The claim then follows from Theorem 1.5. □

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point. Such graphs are widely studied; see [22, 31, 36, 39, 23] for example. For an integer $\delta \geqslant 1$, if each curve is involved in at most δ intersections with other curves, then the corresponding string graph is called a *δ-string graph*.

Corollary 1.8. — *There is a function f such that every δ-string graph G is contained in $J \boxtimes P \boxtimes K_{f(\delta)}$ for some graph J with $\operatorname{tw}(J) \leqslant 15\,288\,899$ and for some path P.*

Proof. — Hickingbotham and Wood [26] observed that G is a $\lfloor\frac{\delta}{2}\rfloor$-shallow minor of $H \boxtimes K_2$, where H is the planar graph obtained by adding a dummy vertex at each intersection point of two curves (and possibly adding isolated vertices). A close inspection of their proof reveals that each branch set of the model of G in $H \boxtimes K_2$ is a path. So G is a $(\lfloor\frac{\delta}{2}\rfloor, 2)$-shallow minor of $H \boxtimes K_2$. The claim then follows from Theorem 1.5. □

The following graph class was introduced by Angelini, Bekos, Kaufmann, Kindermann and Schneck [1]. A *fan-bundling* of a graph G is an indexed set $\mathcal{E} = (\mathcal{E}_v : v \in V(G))$ where $\mathcal{E}_v$ is a partition of the set of edges in G incident to v. Each element of $\mathcal{E}_v$ is called a *fan-bundle*. For a fan-bundling $\mathcal{E}$ of G, let $G_{\mathcal{E}}$ be the graph whose vertices are the vertices of G and the bundles of $\mathcal{E}$, where vB is an edge of $G_{\mathcal{E}}$ whenever $v \in V(G)$ and $B \in \mathcal{E}_v$, and B_1B_2 is an edge of $G_{\mathcal{E}}$ whenever $vw \in E(G)$ and $vw \in B_1 \in \mathcal{E}_v$ and $vw \in B_2 \in \mathcal{E}_w$. A graph G is *k-fan-bundle planar* if for some fan-bundling $\mathcal{E}$ of G, the graph $G_{\mathcal{E}}$ has a drawing in the plane such that each edge $B_1B_2 \in E(G_{\mathcal{E}})$ is in no crossings, and each edge $vB \in E(G_{\mathcal{E}})$ is in at most k crossings.

Corollary 1.9. — *There is a function f such that every k-fan-bundle planar graph G is contained in $J \boxtimes P \boxtimes K_{f(k)}$ for some graph J with $\operatorname{tw}(J) \leqslant 15\,288\,899$ and for some path P.*

Proof. — Hickingbotham and Wood [26] showed that G is a $(k+1)$-shallow minor of $H \boxtimes K_2$ for some planar graph H. A close inspection of their proof reveals that each branch set of the model of G in $H \boxtimes K_2$ is a rooted subdivided star. So G is a $(k+1, 2)$-shallow minor of $H \boxtimes K_2$. The claim then follows from Theorem 1.5. □

1.2. Other Surfaces

We generalise all of the above results for graphs embeddable on any fixed surface as follows. The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that there is an embedding of G in a surface of Euler genus g; see [33] for background about graph embeddings in surfaces. Theorem 1.5 generalises as follows.

Theorem 1.10. — *There is a function f such that for every graph G of Euler genus g, every (r, s)-shallow minor H of $G \boxtimes K_d$ is contained in $J \boxtimes P \boxtimes K_{f(d,r,s,g)}$ for some graph J with $\operatorname{tw}(J) \leqslant 963\,922\,179$.*

Theorem 1.3 generalises as follows. The proof is directly analogous to the proof of Theorem 1.3, using Theorem 1.10 instead of Theorem 1.5.

Corollary 1.11. — *There is a function f such that for every graph G of Euler genus g and for any integers $k, d \geqslant 1$, the graph G^k_d is contained in $H \boxtimes P \boxtimes K_{f(d,g,k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 963\,922\,179$ and for some path P.*

We generalise Corollary 1.7 as follows, where a graph G is *(g, k)-planar* if G has a drawing in a surface of Euler genus g in which each edge is involved in at most k crossings, where no three edges cross at a single point. Such graphs are widely studied [24, 11, 17, 18]. Dujmović *et al.* [18] proved that every (g, k)-planar graph is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}(6k^2+16k+10)}$, for some graph H of treewidth $\binom{k+4}{3} - 1$ and for some path P. We improve the treewidth bound to an absolute constant. The proof is directly analogous to the proof of Corollary 1.7.

Corollary 1.12. — *There is a function f such that every (g, k)-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(g,k)}$ for some graph H with $\operatorname{tw}(H) \leqslant 963\,922\,179$.*

We generalise Corollary 1.8 as follows, where a *(g, δ)-string graph* is the intersection graph of a set of curves in a surface of Euler genus g, such that no three curves meet at a single point, and each curve is involved in at most δ intersections with other curves. The proof of Corollary 1.13 is directly analogous to the proof of Corollary 1.8.

Corollary 1.13. — *There is a function f such that every (g, δ)-string graph G is contained in $J \boxtimes P \boxtimes K_{f(\delta)}$ for some graph J with $\operatorname{tw}(J) \leqslant 963\,922\,179$ and for some path P.*

We generalise Corollary 1.9 as follows, where a graph G is *(g,k)-fan-bundle planar* if for some fan-bundling $\mathcal{E}$ of G, the graph $G_{\mathcal{E}}$ has a drawing in a surface of Euler genus g such that each edge $B_1B_2 \in E(G_{\mathcal{E}})$ is in no crossings, and each edge $vB \in E(G_{\mathcal{E}})$ is in at most k crossings. The proof of Corollary 1.14 is directly analogous to the proof of Corollary 1.9.

Corollary 1.14. — *There is a function f such that every (g,k)-fan-bundle planar graph G is contained in $J \boxtimes P \boxtimes K_{f(k)}$ for some graph J with $\operatorname{tw}(J) \leqslant 963\,922\,179$ and for some path P.*

1.3. Application: Centred Colourings

Nešetřil and Ossona de Mendez [34] introduced the following definition. For an integer $p \geqslant 1$, a vertex colouring ϕ of a graph G is *p-centred* if, for every connected subgraph $X \subseteq G$, $|\{\phi(v) : v \in V(X)\}| > p$ or there exists some $v \in V(X)$ such that $\phi(v) \neq \phi(w)$ for every $w \in V(X) \setminus \{v\}$. For an integer $p \geqslant 1$, the *p-centred chromatic number* of a graph G, denoted by $\chi_p(G)$, is the minimum integer $c \geqslant 0$ such that G has a p-centred c-colouring. Centred colourings are important within graph sparsity theory as they characterise graph classes with bounded expansion [34].

Dębski, Felsner, Micek and Schröder [5] established that $\chi_p(G \boxtimes H) \leqslant \chi_p(G)\chi(H^p)$ for all graphs G and H. Pilipczuk and Siebertz [37, Lemma 15] showed that $\chi_p(G) \leqslant \binom{p+t}{t}$ for every graph G with treewidth at most t. It follows that if $G \subseteq H \boxtimes P \boxtimes K_\ell$ and $\operatorname{tw}(H) \leqslant t$, then

$$\chi_p(G) \leqslant \ell(p+1)\chi_p(H) \leqslant \ell(p+1)\binom{p+t}{t} \in O_\ell(p^{t+1}). \tag{1.1}$$

Thus, Theorem 1.3 and Corollarys 1.7 and 1.9 imply:

- for every planar graph G and any integers $k, d \geqslant 1$, $\chi_p(G^k_d) \in O_{k,d}(p^{15\,288\,900})$;
- for every k-planar graph G, $\chi_p(G) \in O_k(p^{15\,288\,900})$;
- for every k-fan-bundle graph G, $\chi_p(G) \in O_k(p^{15\,288\,900})$.

Similarly, Corollarys 1.11, 1.12 and 1.14 imply:

- for every graph G of Euler genus g and for any integers $k, d \geqslant 1$, $\chi_p(G^k_d) \in O_{g,k,d}(p^{963\,922\,180})$;
- for every (g,k)-planar graph G, $\chi_p(G) \in O_{g,k}(p^{963\,922\,180})$;
- for every (g,k)-fan-bundle graph G, $\chi_p(G) \in O_{g,k}(p^{963\,922\,180})$.

For k-planar or (g,k)-planar graphs G, the best previously known bound was $\chi_p(G) \in O_{g,k}(p^{\binom{k+4}{3}})$, due to Dujmović *et al.* [18]. The above results significantly improve this bound (for large k).

1.4. Paper Outline

It remains to prove Theorems 1.5 and 1.10. The proofs of these results depend on the notion of a 'blocking partition', which we believe is of independent interest. Following a section of preliminary definitions, Section 3 introduces and states our main results about blocking partitions: Theorem 3.1 for planar graphs and Theorem 3.2 for graphs of Euler genus g. We then show how Theorems 3.1 and 3.2 imply Theorems 1.5 and 1.10. Theorem 3.1 is the heart of the paper, and is proved in Sections 4–6. Theorem 3.2 is then proved in Section 7 as a corollary of Theorem 3.1. Section 8 considers which graph classes admit blocking partitions of bounded width. We show that bounded maximum degree is necessary but not sufficient, and that bounded maximum degree and bounded treewidth are sufficient. Section 9 concludes by introducing some natural open problems that arise from this work.

2. Preliminaries

We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. See [7] for graph-theoretic definitions not given here. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$. A *graph class* is a collection of graphs closed under isomorphism.

We use the following notation for a graph G. For $v \in V(G)$, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ and $N_G[v] := N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, let $N_G(S) := \bigcup_{v\in S} N_G(v) \setminus S$.

The *length* of a path P is the number of edges in P. Given a graph G and two subsets $A, B \subseteq V(G)$, a path P in G is an *A–B path* if either P consists of only one vertex $x \in A \cap B$, or P has length at least 1, one end of P belongs to A, the other belongs to B, and no inner vertex belongs to $A \cup B$. For vertices $x, y \in V(G)$, an *x–y path* is an $\{x\}$–$\{y\}$ path. For a tree T and $x, y \in V(T)$, we denote the unique x–y path in T by xTy.

For two subsets $U_1, U_2 \subseteq V(G)$, let $\text{dist}_G(U_1, U_2)$ denote the *distance* between U_1 and U_2 in G; that is, the length of a shortest U_1–U_2 path in G (or $+\infty$ if no such path exists). In this notation, the role of U_i can be played by a vertex x, which is then interpreted as the singleton $\{x\}$; for example, we write $\text{dist}_G(x, U)$ rather than $\text{dist}_G(\{x\}, U)$. Similarly, the role of U_i can be played by an edge $x_1x_2 \in E(G)$, which is then interpreted as the set $\{x_1, x_2\}$, or by a set of edges $M \subseteq E(G)$ which is interpreted as $\bigcup_{xy\in M}\{x, y\}$. A path P in a graph G is *geodesic* if it is a shortest path between its ends in G, which implies $\mathrm{dist}_P(x, y) = \mathrm{dist}_G(x, y)$ for any $x, y \in V(P)$.

In a plane embedding of a graph G, a *face* is a connected component of $\mathbb{R}^2 - G$. We use *closure* and *boundary* in the topological sense. So the closure of a face f is the union of f and the boundary of f.

A *tree-decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of a tree T such that (a) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a non-empty (connected) subtree of T. Each set W_x in $\mathcal{W}$ is called a *bag*. The *width* of $\mathcal{W}$ is $\max\{|W_x| : x \in V(T)\} - 1$. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G. Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth at most 1 if and only if it is a tree.

Let G and H be graphs. A *partition* of G is a collection $\mathcal{P}$ of sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. Empty parts are allowed. The *width* of $\mathcal{P}$ is the maximum number of vertices in a part. The *quotient* of $\mathcal{P}$ (with respect to G) is the graph, denoted by $G/\mathcal{P}$, whose vertices are the non-empty parts of $\mathcal{P}$, where distinct non-empty parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. The quotient is defined analogously when $\mathcal{P}$ is a set of vertex-disjoint subgraphs of G whose vertex-sets partition G. Then the vertices of $G/\mathcal{P}$ are subgraphs of G instead of sets of vertices. An *H-partition* of G is a partition $\mathcal{P}$ of G such that $G/\mathcal{P}$ is contained in H. The following observation connects partitions and products.

Observation 2.1 ([16]). — *For all graphs G and H and any integer $p \geqslant 1$, G is contained in $H \boxtimes K_p$ if and only if G has an H-partition with width at most p.*

A partition of a graph G is *connected* if the subgraph induced by each part is connected. In this case, the quotient is the minor of G obtained by contracting each part into a single vertex.

A partition $\mathcal{P}$ of G is *chordal* if $\mathcal{P}$ is connected and $G/\mathcal{P}$ is *chordal*.

A *tree-partition* is a T-partition for some tree T. Such a T-partition is *rooted* if T is rooted.

Let G and H be graphs and let Z be a subgraph of $G \boxtimes H$. The *projection* of Z onto G is the subgraph Z' of G where

$$V(Z') := \{v \in V(G) \colon (v,x) \in V(Z) \text{ for some } x \in V(H)\} \text{ and}$$
$$E(Z') := \{uv \in E(G) \colon (u,x)(v,y) \in E(Z) \text{ for some } x,y \in V(H)\}.$$

A *BFS-layering* of a connected graph G is an ordered partition $(V_0, V_1, \dots)$ of $V(G)$ where $V_0 = \{r\}$ for some vertex $r \in V(G)$ and $V_i = \{v \in V(G) : \text{dist}_G(v,r) = i\}$ for each $i \geqslant 1$. A path P is *vertical* with respect to $(V_0, V_1, \dots)$ if $|V(P) \cap V_i| \leqslant 1$ for all $i \geqslant 0$. Let T be a spanning tree of G, where for each non-root vertex $v \in V_i$ there is a unique edge vw in T for some $w \in V_{i-1}$. Then T is called a *BFS-spanning tree* of G.

3. Blocking Partitions

Let G be a graph and $\mathcal{R}$ be a connected partition of $V(G)$. A path P in G is *$\mathcal{R}$-clean* if $|V(P) \cap V| \leqslant 1$ for each part $V \in \mathcal{R}$. We say that $\mathcal{R}$ is *ℓ-blocking* if every $\mathcal{R}$-clean path in G has length at most ℓ, as illustrated in Figure 3.1.

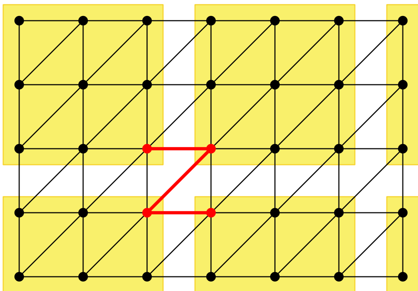


Figure 3.1. A 3-blocking partition $\mathcal{R}$ of width 9. The red path is a longest $\mathcal{R}$-clean path.

The following result is the heart of this paper.

Theorem 3.1. — *Every planar graph G with maximum degree Δ has a 222-blocking partition $\mathcal{R}$ with width at most* $f(\Delta) := 10\Delta^{80}(3612\,\Delta^{452} + 900)$.

Theorem 3.1 is proved in Sections 4–6. Section 7 proves the following extension of Theorem 3.1.

THEOREM 3.2. — *Every graph G with Euler genus g and maximum degree Δ has a 894-blocking partition with width at most $f(\Delta, g) := \max\{10\Delta^{80}(3612\Delta^{452} + 900), 8950g^2 + 1796g\}$.*

To show that Theorems 3.1 and 3.2 imply our main results (Theorems 1.5 and 1.10), we use the following lemma.

LEMMA 3.3. — *Let $\mathcal{G}$ be a minor-closed class such that for some function f and integers $\ell, t, c \geqslant 1$,*

- *every graph $G \in \mathcal{G}$ has an ℓ-blocking partition $\mathcal{R}$ with width at most $f(\Delta(G))$;*
- *every graph $G \in \mathcal{G}$ is contained in $H \boxtimes P \boxtimes K_c$ for some graph H with $\operatorname{tw}(H) \leqslant t$ and for some path P.*

Then there is a function g such that for any integers $r \geqslant 0$ and $d, s \geqslant 1$, for every graph $G \in \mathcal{G}$, every (r, s)-shallow minor of $G \boxtimes K_d$ is contained in $J \boxtimes P \boxtimes K_{g(d,r,s,\ell,c)}$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{2\ell+5+t}{t} - 1$ and for some path P.

For planar graphs, Lemma 3.3 is applicable with $\ell = 222$ by Theorem 3.1 and with $t = c = 3$ by Theorem 1.1. Lemma 3.3 thus proves Theorem 1.5 since $\operatorname{tw}(J) \leqslant \binom{2\ell+5+t}{t} - 1 = \binom{2\cdot 222+5+3}{3} - 1 = 15\,288\,899$. For graphs of Euler genus g, Lemma 3.3 is applicable with $\ell = 894$ by Theorem 3.2 and with $t = 3$ and $c = \max\{2g, 3\}$ by a result of Distel, Hickingbotham, Huynh and Wood [9]. Lemma 3.3 thus proves Theorem 1.10 since $\operatorname{tw}(J) \leqslant \binom{2\ell+5+t}{t} - 1 = \binom{2\cdot 894+5+3}{3} - 1 = 963\,922\,179$.

We now work towards proving Lemma 3.3.

LEMMA 3.4. — *Let $\mathcal{G}$ be a minor-closed class such that, for some function f and integer $\ell \geqslant 1$, every graph $G_0 \in \mathcal{G}$ has an ℓ-blocking partition $\mathcal{R}$ with width at most $f(\Delta(G_0))$. Then for any integers $r > \ell + 2$ and $s, d \geqslant 1$, for every graph $G \in \mathcal{G}$, every (r, s)-shallow minor H of $G \boxtimes K_d$ is an $(r-1, s')$-shallow minor of $G' \boxtimes K_{d'}$, where G' is a minor of G and is thus in $\mathcal{G}$, and $s' = (ds)^r$ and $d' = d \cdot f(ds)$.*

Proof. — Let $((B_x, v_x) \colon x \in V(H))$ be an (r, s)-shallow model of H in $G \boxtimes K_d$. For each $x \in V(H)$, let B'_x and v'_x be the projections of B_x and v_x, respectively, onto G. Observe that for each $x \in V(H)$, the maximum degree of each $B_x - v_x$ is at most s and each vertex in B'_x is at distance at most r from v'_x. Let $G_0 := \bigcup(B'_x - v'_x : x \in V(H))$, which is a subgraph of G and therefore in $\mathcal{G}$. Since every vertex in G_0 has at most d vertices mapped to it, the maximum degree of G_0 is at most ds. By assumption, there is an ℓ-blocking partition $\mathcal{R}$ of G_0 with width at most $f(ds)$.

Let $\mathcal{R}' := \mathcal{R} \cup \{\{v\} : v \in V(G) \setminus V(G_0)\}$, which is a partition of G. Define $G' := G/\mathcal{R}'$. Since $\mathcal{R}'$ is a connected partition, G' is a minor of G and is therefore in $\mathcal{G}$. The width of $\mathcal{R}'$ is at most $f(ds)$, so G is contained in $G' \boxtimes K_{f(ds)}$ by Observation 2.1. By slightly abusing the notation, we identify the graph G with the isomorphic subgraph of $G' \boxtimes K_{f(ds)}$. So the graphs B'_x are subgraphs of $G' \boxtimes K_{f(ds)}$, and each vertex of $G' \boxtimes K_{f(ds)}$ belongs to at most d graphs B'_x.

For each $x \in V(H)$, let T'_x be a BFS-spanning tree of B'_x rooted at v'_x. Hence, the maximum degree of $T'_x - v'_x$ is at most ds and each vertex is at distance at most r from the root v'_x in T'_x. So each component of $T'_x - v'_x$ has at most $(ds)^0 + \ldots + (ds)^{r-1} < (ds)^r$ vertices'. Let $\overline{T'_x}$ denote the graph obtained from T'_x by adding each edge of $G' \boxtimes K_{f(ds)}$ that joins a vertex of T'_x to one of its descendants. Then $\overline{T'_x} - v'_x$ has maximum degree at most $(ds)^r$.

Below we show that the maximum degree of $\overline{T'_x} - v'_x$ is at most s' and each vertex in $\overline{T'_x}$ is at distance at most $r-1$ from v'_x. This implies that H is an $(r-1, s')$-shallow minor of $G' \boxtimes K_{f(ds)} \boxtimes K_d$, where an appropriate model can be defined by choosing for each $v \in V(G' \boxtimes K_{f(ds)})$ an injective map from $\{x \in V(H) : v \in V(B'_x)\}$ to $V(K_d)$. Since $G' \boxtimes K_{f(ds)} \boxtimes K_d$ is isomorphic to $G' \boxtimes K_{d'}$, the lemma will follow.

First we estimate the maximum degree of $\overline{T'_x} - v'_x$. Consider a vertex $v \in V(T'_x)$ at distance $i \geqslant 1$ from the root v'_x in T'_x. Then, for each $j \in \{1, \ldots, i-1\}$, the vertex v has only one ancestor at distance j from v'_x in T'_x. Since the maximum degree of $T'_x - v'_x$ is at most ds, for each $j \in \{i+1, \ldots, r\}$, there are at most $(ds)^{j-i}$ descendants of v at distance j from v'_x. Therefore, v has at most $(ds)^{j-1}$ neighbours in $\overline{T'_x} - v'_x$ which are at distance j from v'_x in T'_x. Hence, the degree of v in $\overline{T'_x} - v'_x$ is at most $(ds)^0 + \ldots + (ds)^{r-1} < (ds)^r$, so the maximum degree of $\overline{T'_x} - v'_x$ is at most $(ds)^r$.

It remains to show that in each $\overline{T'_x}$, every vertex is at distance at most $r-1$ from v'_x. Suppose to the contrary that some vertex u is at distance at least r from v'_x in $\overline{T'_x}$. Since $T'_x \subseteq \overline{T'_x}$, and in T'_x every vertex is at distance at most r from v'_x, the vertex u must be at distance exactly r from v'_x in T'_x and $\overline{T'_x}$. Let $P = (u_0, \ldots, u_r)$ be the unique path between v'_x and u in T'_x where $u_0 = v'_x$ and $u_r = u$. Let $P' = (x_1, \ldots, x_r)$ be the projection of P onto G_0. Then P' is a path in G_0 with length $r-1 \geqslant \ell+1$, so it contains two vertices x_α and x_β with $1 \leqslant \alpha < \beta$ that belong to the same part in $\mathcal{R}$. Thus the projection of u_α and u_β (in $G' \boxtimes K_{f(ds)}$) are the same vertex in G' and so, by the definition of the strong product, $u_\beta u_{\alpha-1} \in E(G' \boxtimes K_{f(ds)})$. Hence the distance between v'_x and u in $\overline{T'_x}$ is less than r, a contradiction. □

We prove the next lemma by iteratively applying Lemma 3.4.

Lemma 3.5. — *Let $\mathcal{G}$ be a minor-closed class such that, for some function f and integer $\ell \geqslant 1$, every graph $G \in \mathcal{G}$ has an ℓ-blocking partition with width at most $f(\Delta(G))$. Then there is a function h such that for any integers $r \geqslant 0$ and $s, d \geqslant 1$, for every graph $G \in \mathcal{G}$, every (r,s)-shallow minor H of $G \boxtimes K_d$ is an $(\ell+2)$-shallow minor of $Q \boxtimes K_{h(d,r,s,\ell)}$ for some minor Q of G.*

Proof. — If $r \leqslant \ell+2$ then H is an $(\ell+2)$-shallow minor of $Q \boxtimes K_{h(d,r,s,\ell)}$, where $Q = G$ and $h(d,r,s,\ell) = d$, and we are done. Now assume that $r > \ell+2$. Thus $r-\ell-2 \geqslant 1$. Let $d_0 := d$ and $s_0 := s$. Iteratively applying Lemma 3.4, we obtain a sequence $G_1, G_2, \dots, G_{r-\ell-2}$ of minors of G, such that for each $i \in \{1,\dots,r-\ell-2\}$, H is an $(r-i, s_i)$-shallow minor of $G_i \boxtimes K_{d_i}$, where $s_i = (d_{i-1}s_{i-1})^{r-i+1}$ and $d_i = d_{i-1} \cdot f(d_{i-1}s_{i-1})$. In particular (with $i = r-\ell-2$), H is an $(\ell+2)$-shallow minor of $G_{r-\ell-2} \boxtimes K_{d_{r+\ell-2}}$. The result follows with $Q := G_{r-\ell-2}$ and $h(d,r,s,\ell) := d_{r-\ell-2}$. □

Proof of Lemma 3.3. — Let $G \in \mathcal{G}$ and let G' be an (r,s)-shallow minor of $G \boxtimes K_d$. By Lemma 3.5, G' is an $(\ell+2)$-shallow minor of $Q \boxtimes K_{h(d,r,s,\ell)}$ for some minor Q of G. Thus $Q \in \mathcal{G}$. By assumption, Q is contained in $H \boxtimes P \boxtimes K_c$ for some graph H with $\operatorname{tw}(H) \leqslant t$. Hence G' is an $(\ell+2)$-shallow minor of $H \boxtimes P \boxtimes K_{c\,h(d,r,s,\ell)}$. By Theorem 1.4, G' is contained in $J \boxtimes P \boxtimes K_{c(2(\ell+2)+1)^2 \cdot g(d,r,s,\ell)}$ for some graph J with $\operatorname{tw}(J) \leqslant \binom{2(\ell+2)+1+t}{t} - 1$. The result follows with $g(d,r,s,\ell,c) := c(2(\ell+2)+1)^2 \cdot h(d,r,s,\ell)$. □

4. The Chordal Partition

Our focus now is the proof of Theorem 3.1, which is inspired by the construction of a chordal partition of a planar triangulation by van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [25]. They showed that every planar triangulation G has a partition $\mathcal{P}$ into paths $P_1, \dots, P_n$, such that for each $i \in \{1,\dots,n-1\}$, the path P_{i+1} is geodesic in $G - (V(P_1) \cup \dots \cup V(P_i))$, P_{i+1} is adjacent to at most two of the paths $P_1, \dots, P_i$, and if P_{i+1} is adjacent to P_a and P_b with $1 \leqslant a < b \leqslant i$, then P_a is adjacent to P_b. Then the quotient $G/\mathcal{P}$ is chordal with treewidth 2.

Our ℓ-blocking partition of a planar graph G will be obtained from a partition of G into subtrees $T_1, \dots, T_n$ with similar properties: for each

$i \in \{1, \dots, n-1\}$, the tree T_{i+1} is adjacent to at most two of the trees $T_1, \dots, T_i$, and if T_{i+1} is adjacent to two of those trees, then they are adjacent to each other. The final partition is then obtained by appropriately breaking each $V(T_i)$ into connected parts of bounded size.

Fix a planar graph G of maximum degree Δ and any planar embedding of G. This section constructs a 6-blocking[(4)] chordal partition $\mathcal{T}$ of G. Later sections refine this partition into a connected (non-chordal) partition $\mathcal{R}$ with width bounded in terms of Δ, and show that $\mathcal{R}$ is 222-blocking, which will prove Theorem 3.1. Since Theorem 3.1 is trivial when $\Delta \leqslant 2$, we assume that $\Delta \geqslant 3$.

Our construction of the 6-blocking chordal partition is parameterised by a positive integer τ. Ultimately, we will fix $\tau = 37$, but it will be easier to visualise the construction for smaller values of τ.

We use the notion of F-bridges, as illustrated in Figure 4.1. For a subgraph F of G, an *F-bridge* is either a length-1 path in G that is edge-disjoint from F and is between two vertices in $V(F)$ (such an F-bridge is *trivial*), or a graph obtained from a component C of $G - V(F)$ by adding all vertices in $N_G(V(C))$ and all edges of G between $V(C)$ and $N_G(V(C))$ (such an F-bridge is *non-trivial*). Observe that each edge of G outside F belongs to exactly one F-bridge. In an F-bridge B, the set $V(B) \cap V(F)$ is the *attachment-set*, and its elements are the *attachment-vertices* of B. Hence, if B is non-trivial with attachment-set A, then $B - A$ is a component of $G - V(F)$.

We will inductively define a sequence of vertex-disjoint trees $T_1, \dots, T_m$ whose vertex-sets will form the chordal partition of G. For each $j \in \{0, \dots, m\}$, we denote the forest $\bigcup_{i<j} T_i$ by F_j (in particular, F_0 is empty). For each $j \in \{0, \dots, m\}$, we maintain the following invariant.

Invariant. — For every non-trivial F_j-bridge B:

(i) B has attachment-vertices on at most two components of F_j;

(ii) for every component T_i of F_j that contains an attachment-vertex of B, the tree T_i is contained in the closure of the outer-face of B, and the attachment-vertices of B in $V(T_i)$ are leaves of T_i;

(iii) if B has attachment-vertices on two distinct components T_i and $T_{i'}$ of F_j, then T_i is contained in the closure of the outer-face of $T_{i'} \cup B$, and $T_{i'}$ is contained in the closure of the outer-face of $T_i \cup B$.

This invariant implies the following.

(4) Actually, the partition is 4-blocking, but for simplicity we prove a weaker bound.

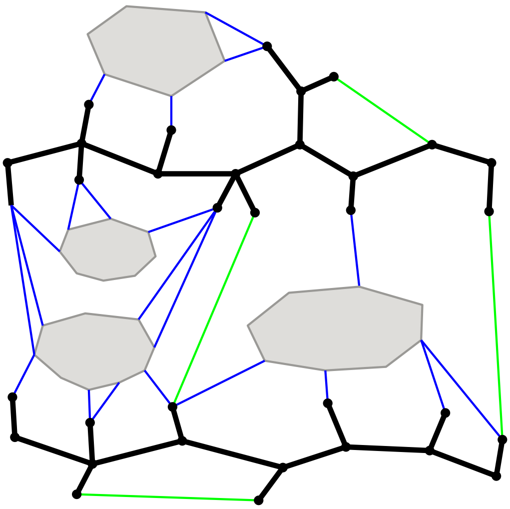


Figure 4.1. A graph G with a distinguished sub-forest F with two components (black). There are four trivial F-bridges (green) and four non-trivial F-bridges (each of them is obtained from a component C of $G - V(F)$ (gray) by adding all blue edges incident with a vertex of C (and their ends outside C))

Claim 1. — *Suppose that the invariant is satisfied for some $j \in \{0, \ldots, m\}$, and let B be a nontrivial F_j-bridge with at least one attachment-vertex. Let J be the union of B and all components T_i of F_j that contain an attachment-vertex of B. Then, for each component T_i contained in J, at least one and at most two attachment-vertices of B on T_i are on the boundary of the outer-face of J.*

Proof. — By invariant (i), B has attachment-vertices on at most one component of F_j distinct from T_i. By invariant (ii), T_i is contained in the closure of the outer-face of B. Moreover, by invariant (iii), if B has an attachment-vertex on a tree $T_{i'}$ distinct from T_i, then T_i is contained in the closure of the outer-face of $B \cup T_{i'}$. Therefore, in the facial walk along the outer-face of J, the vertices and edges that belong to T_i appear consecutively, forming a (possibly closed) sub-walk W. By invariant (ii), the attachment-vertices of B in $V(T_i)$ are leaves of T_i, so only the terminal vertices of W are attachments of B in $V(T_i)$ which lie on the boundary of the outer-face of J. At most two vertices are terminal vertices of W, so the claim holds. □

For $j = 0$, the invariant is satisfied because F_0 is empty, so the F_0-bridges have no attachment-vertices.

Together with each tree T_j we will define a tuple $(B_j, A_j, X_j, A_j^{\text{out}}, D_j, T_j^0)$, where $T_j^0 \subseteq T_j \subseteq B_j \subseteq G$, $A_j^{\text{out}} \subseteq A_j \subseteq V(F_{j-1})$, $X_j \subseteq \{1, \dots, j-1\}$, and $D_j \subseteq V(T_j^0)$.

Let $j \geqslant 1$ be an integer, and suppose that we have already defined the trees $T_1, \dots, T_{j-1}$, and thus the forest F_{j-1} is defined. If $V(F_{j-1}) = V(G)$, then terminate the construction with a sequence of length $j-1$. Otherwise, let B_j be any non-trivial F_{j-1}-bridge. Let A_j denote the attachment-set of B_j, and let X_j denote the set of all $i \in \{1, \dots, j-1\}$ such that B_j has an attachment-vertex in $V(T_i)$. By invariant (i), we have $|X_j| \leqslant 2$.

Let $J := B_j \cup \bigcup_{i \in X_j} T_i$. Define A_j^{out} to be the set of attachment-vertices $x \in A_j$ that lie on the boundary of the outer-face of J. By Claim 1, the set A_j^{out} contains one or two vertices of each T_i with $i \in X_j$, so $|A_j^{\text{out}}| \leqslant 4$ and if $A_j \neq \varnothing$, then $A_j^{\text{out}} \neq \varnothing$.

Define a non-empty subset $D_j \subseteq V(B_j - A_j)$ as follows. If $A_j = \varnothing$, then let D_j be a set consisting of one arbitrary vertex on the boundary of the outer-face of B_j. If $A_j \neq \varnothing$ (and thus $A_j^{\text{out}} \neq \varnothing$), then let D_j denote the set of all vertices $x \in V(B_j - A_j)$ such that $\text{dist}_G(x, A_j^{\text{out}}) \leqslant \tau$. In B_j, every vertex from A_j^{out} has a neighbour in $V(B_j - A_j)$ and such a neighbour belongs to D_j (recall that $\tau \geqslant 1$). Hence, D_j is non-empty.

Let T_j^0 be a tree in $B_j - A_j$ that contains all vertices in D_j and has the smallest possible number of edges, and let T_j be a tree obtained from T_j^0 by attaching each vertex $x \in N_{B_j - A_j}(V(T_j^0))$ with any edge of G between x and $V(T_j^0)$. See Figure 4.2.

We now verify that for such T_j, the invariant is satisfied. Let B be a non-trivial F_j-bridge. If B has no attachment-vertex on T_j, then B is an F_{j-1}-bridge distinct from B_j, and the invariant is satisfied by induction. Hence, we assume that B has an attachment-vertex on T_j. Since every component of $G - V(F_j)$ which is adjacent in G to T_j is a component of $(B_j - A_j) - V(T_j)$, the F_j-bridge B is contained in B_j. Note that every attachment-vertex of B that is not on T_j lies on a tree T_i with $i < j$, and thus, is an attachment-vertex of B_j.

Suppose first that $X_j = \varnothing$. Then B has all its attachment-vertices on T_j. The only vertex x of D_j is on the boundary of the outer-face of B_j. The tree T_j is a star with a centre at x and whose leaves are the neighbours of x in G. Therefore B can intersect T_j only in its leaves, and T_j is in the closure of the outer-face of B, so the invariant holds.

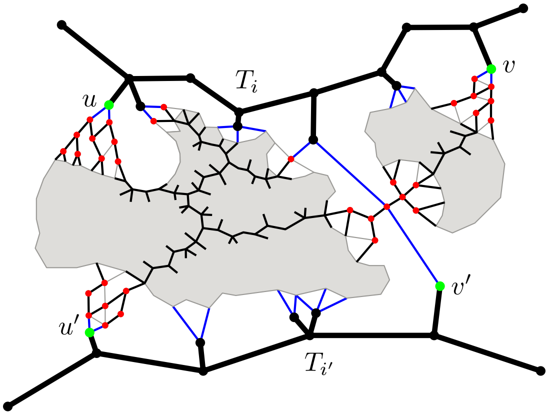


Figure 4.2. A possible situation in the construction of the tree T_j for $\tau = 3$, where B_j has attachment-vertices on the trees T_i and $T_{i'}$. Here, $A_j^{\text{out}} = \{u, v, u', v'\}$, the vertices from D_j are red, and the tree T_j consists of the red and black edges .

Now suppose that $X_j \neq \varnothing$, and let $i \in X_j$. By induction, the tree T_i intersects the boundary of the outer-face of J, and we can write the facial walk along the outer-face of J as $W = v_0e_0v_1e_1 \cdots e_{n-1}v_n$ where $v_0 = v_n$ and for some $s \in \{0, \dots, n-1\}$ we have $V(W) \cap V(T_i) = \{v_0, \dots, v_s\}$ and $E(W) \cap E(T_i) = \{e_0, \dots, e_{s-1}\}$. We have $A_j^{\text{out}} \cap V(T_i) = \{v_0, v_s\}$ (possibly $v_0 = v_s$). Each of the edges $e_{n-1} = v_0v_{n-1}$ and $e_s = v_sv_{s+1}$ has an end in $V(T_i)$ but does not belong to T_i. Hence both of these edges are edges of B_j, so $\{v_{n-1}, v_{s+1}\} \subseteq V(B_j - A_j)$. Since $\{v_0, v_s\} \subseteq A_j^{\text{out}}$, we have $\{v_{n-1}, v_{s+1}\} \subseteq D_j \subseteq V(T_j^0)$.

We now show that B has attachment-vertices on at most two components of F_j. Every attachment-vertex of B that is not in $V(T_j)$, is an attachment-vertex of B_j. Hence, if $X_j = \{i\}$, then B can only have attachment-vertices on T_j and T_i. Therefore, suppose that $X_j = \{i, i'\}$ with $i' \neq i$. We need to show that B has an attachment-vertex on at most one of the trees T_i and $T_{i'}$. By our invariant, $T_{i'}$ is in the closure of the outer-face of $T_i \cup B$. Therefore, $T_{i'}$ intersects $\{v_{s+2}, \dots, v_{n-2}\}$. Since the vertices $v_0, \dots, v_s$ belong to T_i, the path $v_{n-1}T_j^0v_{s+1}$ separates the trees T_i and $T_{i'}$ in J. Since $T_j^0 \subseteq T_j$, every component of $(B_j - A_j) - V(T_j)$ is adjacent to at most one of the trees T_i and $T_{i'}$. Since B is a non-trivial F_j-bridge contained in B_j, this means that B has attachment-vertices in at most one of the trees

T_i and $T_{i'}$, as required. Hence, B has attachment-vertices on at most two components of F_j.

Assume without loss of generality that every attachment-vertex of B that does not lie on T_j belongs to T_i.

Next, we show that every attachment-vertex of B is a leaf of T_i or T_j. Every attachment-vertex of B on T_i is an attachment-vertex of B_j, and by induction, is a leaf of T_i. Since the tree T_j was obtained from T_j^0 by attaching all adjacent vertices in $B_j - A_j$ as leaves, all attachment-vertices of B on T_j belong to $V(T_j) \setminus V(T_j^0)$, and therefore are leaves of T_j.

Finally, we argue that the tree T_i is in the closure of the outer-face of $T_j \cup B$, and the tree T_j is in the closure of the outer-face of $T_i \cup B$. This will imply that the trees T_i and T_j are in the closure of the outer-face of B, which will complete the proof of the invariant. By induction, the tree T_i is in the closure of the outer-face of B_j. Since $T_j \cup B \subseteq B_j$, the tree T_i is in the closure of the outer-face of $T_j \cup B$. The vertex $v_{s+1} \in V(T_j^0)$ is on the boundary of the outer-face of J. Since T_j^0 and $T_i \cup B$ are disjoint subgraphs of J, the tree T_j^0 is on the outer-face of $T_i \cup B$. The tree T_j is obtained from T_j^0 by attaching leaves, so it is contained in the closure of the outer-face of $T_i \cup B$. This completes the proof of the invariant and the inductive construction.

From now on, we assume that $\mathcal{T} = \{T_1, \ldots, T_m\}$ is a fixed partition obtained by our construction for $\tau = 37$, with a tuple $(B_j, A_j, X_j, A_j^{\text{out}}, D_j, T_j^0)$ associated to each tree T_j.

For later reference, we make explicit some implications of the inductive construction.

Claim 2. — *For each $j \in \{1, \ldots, m\}$, if $X_j = \{i, i'\}$ with $i \neq i'$, then the tree T_j^0 separates the trees T_i and $T_{i'}$ in the graph $T_i \cup T_{i'} \cup B_j$. Consequently, the tree T_j^0 separates the sets $A_j \cap V(T_i)$ and $A_j \cap V(T_{i'})$ in the graph B_j.*

Claim 3. — *For each $j \in \{1, \ldots, m\}$, no non-trivial F_j-bridge has an attachment-vertex on T_j^0.*

Claim 4. — *For any $j \in \{1, \ldots, m\}$ and $i \in X_j$, the graph B_j contains an edge between $A_j^{\text{out}} \cap V(T_i)$ and D_j. In particular, T_i is adjacent to T_j in G.*

We will also use the following simple properties of our construction.

Claim 5. — *For $j \in \{1, \ldots, m\}$, an F_j-bridge B has an attachment-vertex on T_j if and only if $B \subseteq B_j$.*

Proof. — Suppose that B has an attachment-vertex on T_j. If B is trivial, then it consists of one edge with an end in the component $B_j - A_j$ of

$G - V(F_{j-1})$, and hence $B \subseteq B_j$. If B is non-trivial, then it is obtained by adding attachment-vertices to a component of $G - V(F_j)$ adjacent to T_j. That component is contained in $B_j - A_j$, so again $B \subseteq B_j$.

Now suppose that $B \subseteq B_j$. If B is trivial, then its two attachment-vertices belong to $A_j \cup V(T_j)$. Since A_j is an independent set in B_j, at least one attachment-vertex of B is on T_j. If B is non-trivial, then since $B \subseteq B_j$, it is obtained from a component of $(B_j - A_j) - V(T_j)$ by adding all vertices adjacent to it as attachment-vertices. Since $B_j - A_j$ is connected, at least one of these attachment-vertices will lie on T_j. □

Claim 6. — *For $j \in \{1, \dots, m\}$, every non-trivial F_j-bridge B is equal to B_k for some $k \in \{j+1, \dots, m\}$.*

Proof. — The vertex-sets of the trees $T_1, \dots, T_m$ partition $V(G)$, so there exists the least $k \in \{j+1, \dots, m\}$ that contains a non-attachment-vertex of B. Hence, B intersects F_{k-1} only in its attachment-vertices, and they all belong to F_j, so B is an F_{k-1}-bridge that intersects T_k, and therefore $B = B_k$. □

Although we do not use this in our proof, we now show that $\mathcal{T}$ is a chordal partition.

Claim 7. — *$\mathcal{T}$ is a chordal partition with* $\operatorname{tw}(G/\mathcal{T}) \leqslant 2$.

Proof. — Clearly $\mathcal{T}$ is a connected partition since each part has a spanning subtree. Let $j \in \{1, \dots, m\}$. If T_j is adjacent in G to a tree T_i with $i < j$, then, since $T_j \subseteq B_j - A_j$, the F_{j-1}-bridge B_j has an attachment-vertex on T_i, that is, $i \in X_j$. Since $|X_j| \leqslant 2$, the tree T_j can be adjacent to at most two of the trees $T_1, \dots, T_{j-1}$. It remains to show that if T_j is adjacent to two trees T_i and $T_{i'}$ with $i < i' < j$, then T_i is adjacent to $T_{i'}$ in G. Since T_j is adjacent to $T_{i'}$, we have $i' \in X_j$, and therefore, by Claim 5, we have $B_j \subseteq B_{i'}$. The attachment-vertices of B_j on T_i are thus attachment-vertices of $B_{i'}$, so $i \in X_{i'}$. By Claim 4, T_i is adjacent to $T_{i'}$. □

The following property of our chordal partition will play a key role in the proof.

Claim 8. — *Let $j, k \in \{1, \dots, m\}$ be such that B_k is an F_j-bridge contained in B_j, let B be a (possibly trivial) F_j-bridge contained in B_j that is distinct from B_k and has an attachment-vertex in A_j, and let Q be a $V(B)$–$V(B_k)$ path in $B_j - V(T_j^0)$. Then the end of Q in $V(B_k)$ belongs to A_k^{out}.*

Proof. — By Claim 5, each of the F_j-bridges B and B_k has an attachment-vertex on T_j. The F_j-bridge B has attachment-vertices on at

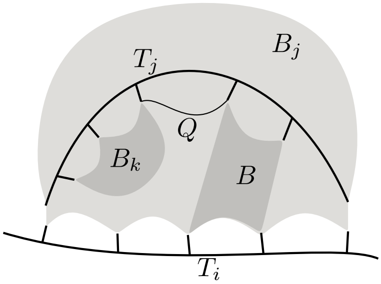


Figure 4.3. Illustration of Claim 8. The end of Q in B_k must belong to A_k^{out}

most two components of F_j, so the attachment-vertices of B in A_j must belong to one tree T_i with $i < j$. We show that every attachement-vertex of B_k that does not lie on T_j belongs to T_i. By Claim 3, the non-trivial F_j-bridge B_k is disjoint from T_j^0. Likewise, if B is non-trivial, then it is disjoint from T_j^0. Otherwise B is trivial and it can have attachments on T_j^0. Let $B' := B - (V(B) \cap V(T_j^0))$. Hence, B' is a connected graph that contains all attachment-vertices of B on T_i and an end of Q. The graph $B_k \cup Q \cup B'$ is therefore a connected subgraph of $B_j - V(T_j^0)$ that intersects T_i. Therefore, by Claim 2, the graph $B_k \cup Q \cup B'$ intersects A_j only in vertices belonging to T_i, so indeed any attachment-vertices of B_k not on T_j must lie on T_i. See Figure 4.3.

We claim that the only face of $T_i \cup T_j \cup B_k$ whose boundary intersects both T_i and T_j is the outer-face. By our invariant, this is true when B_k has attachment-vertices in both T_i and T_j, so suppose that B_k has attachment-vertices only on T_j. Then $T_i \cup T_j \cup B_k$ has two components T_i and $T_j \cup B_k$. The graph T_j is in the closure of the outer-face of B_k, and the graph T_i is in the closure of the outer-face of B_j. Since $T_j \cup B_k \subseteq B_j$, this means that T_i is on the outer-face of $T_j \cup B_k$, and the only face of $T_i \cup T_j \cup B_k$ whose boundary intersects T_i and T_j is the outer-face. Therefore, B is contained in the closure of the outer-face of $T_i \cup T_j \cup B_k$. By Claim 3, B_j is disjoint from T_i^0. Since $Q \subseteq B_j - V(T_j^0)$, this means that Q is disjoint from T_i^0 and T_j^0, and thus Q can intersect the trees T_i and T_j only in their leaves. Furthermore, Q intersects B_k only in one end, so the path Q belongs to the closure of the outer-face of $T_i \cup T_j \cup B_k$ together with B. Hence, the path Q intersects B_k in a vertex on the boundary of the outer-face of $T_i \cup T_j \cup B_k$. That vertex is an attachment-vertex of B_k in A_k^{out}. □

The following claim, while not used in the main proof, provides helpful intuition for the more complicated proof that follows. The proof of this claim does not rely on the value of τ, and even works with $\tau = 1$. Also, the trees T_j^0 do not need to minimise the number of edges for this proof to work. These properties will be useful later, in the proof of Theorem 3.1.

Claim 9. — *The partition $\mathcal{T}$ is 6-blocking.*

Proof. — Consider a $\mathcal{T}$-clean path P in G. We now show that the length of P is at most 6. Let T_i be the tree that intersects P and has the smallest i. Since P is $\mathcal{T}$-clean, it intersects T_i in only one vertex, which splits P into two edge-disjoint paths, each intersecting T_i only in one of its ends. Therefore, it suffices to show that if $Q = (x_0, \ldots, x_p)$ is a $\mathcal{T}$-clean path such that $V(Q) \cap V(F_i) = \{x_0\} \subseteq V(T_i)$, then $p \leqslant 3$.

Suppose towards a contradiction that $p \geqslant 4$. Since $V(Q) \cap V(F_i) = \{x_0\}$, the path Q is contained in a non-trivial F_i-bridge. By Claim 6, that F_i-bridge is equal to B_j for some $j \in \{i+1, \ldots, m\}$. Fix the largest $j \in \{i+1, \ldots, m\}$ such that $Q \subseteq B_j$. We split the argument into two cases based on whether the path Q intersects T_j^0 or not.

Suppose first that $x_\alpha \in V(T_j^0)$ for some $\alpha \in \{1, \ldots, p\}$. Since Q is $\mathcal{T}$-clean, x_α is the only vertex of Q on T_j. In particular, the vertex $x_{\alpha-1}$ is adjacent to T_j^0 in B_j and does not belong to T_j, so $x_{\alpha-1} \in A_j$. The path $x_0 Q x_{\alpha-1}$ is disjoint from $V(T_j^0)$, so by Claim 2 it contains attachment-vertices of B_j on at most one component of F_{j-1}. Since $x_0 \in V(T_i)$ and $x_{\alpha-1} \in A_j$, this implies that $x_{\alpha-1} \in V(T_i)$, and therefore $\alpha - 1 = 0$, that is $\alpha = 1$.

The vertex x_0 is the only vertex of Q on T_i, and the vertex x_1 is the only vertex of Q on T_j. We have $x_1 \in V(T_j^0)$, so by definition of T_j the vertex x_2 is an attachment-vertex of B_j on a tree $T_{i'}$ distinct from T_i. By our choice of i, we have $i < i' < j$. Hence, B_j has attachment-vertices only on T_i and $T_{i'}$. Since $Q \subseteq B_j$, and Q is $\mathcal{T}$-clean, this implies $V(Q) \cap V(F_j) = \{x_0, x_1, x_2\}$. Since $p \geqslant 4$, the path $x_2 Q x_p$ is contained in a non-trivial F_j-bridge which, by Claim 6 is equal to B_k for some $k \in \{j+1, \ldots, m\}$. Since $Q \subseteq B_j$, we have $B_k \subseteq B_j$. The edge $x_1 x_2$ is a trivial F_j-bridge contained in B_k that contains an attachment-vertex in A_j, and its attachment-vertex x_2 belongs to B_k (see Figure 4.4a). Hence, by Claim 8 applied to the trivial path consisting of the vertex x_2 alone, we have $x_2 \in A_k^{\text{out}}$.

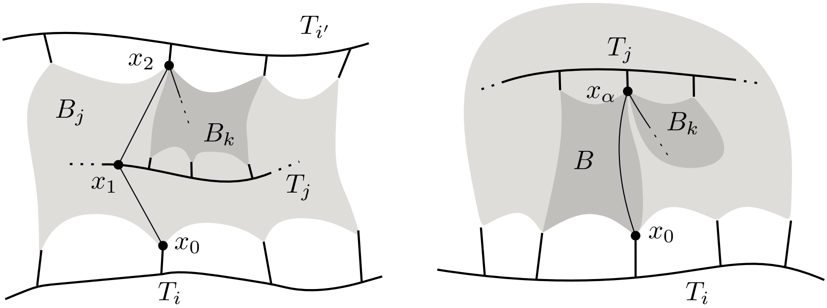
We have $B_k \subseteq B_j$, so by Claim 5, the F_j-bridge B_k has an attachment-vertex on T_j. The vertex x_2 is an attachment-vertex of B_k on $T_{i'}$, so B_k has attachment-vertices only on $T_{i'}$ and T_j. Since $V(Q) \cap V(T_{i'}) = \{x_2\}$ and $V(Q) \cap V(T_j) = \{x_1\}$, we have $x_3 Q x_p \subseteq B_k - A_k$. Since $x_2 \in A_k^{\text{out}}$ this implies that $x_3 \in D_k \subseteq V(T_k^0)$, and thus $x_4 \in V(T_k)$. Hence $\{x_3, x_4\} \subseteq V(T_k)$, contrary to the assumption that Q is $\mathcal{T}$-clean.

Now consider the case when Q is disjoint from T_j^0. We have $x_0 \in V(T_i)$ and $x_1 \notin V(T_i)$, so $x_0x_1 \notin E(F_j)$. Let B be the F_j-bridge containing the edge x_0x_1. Because $x_0x_1 \in E(B_j)$, we have $B \subseteq B_j$, so by Claim 5, B has an attachment-vertex on T_j. The vertex x_0 is an attachment-vertex of B on T_i, so B has attachment-vertices only on T_i and T_j. Observe that $Q \not\subseteq B$; indeed, if B is trivial, this follows from the fact that $p \geqslant 4$, and if B is a non-trivial F_j-bridge, then by Claim 6, we have $B = B_k$ for some $k \in \{j+1, \dots, m\}$, and $Q \not\subseteq B_k$ by our choice of j. Since $Q \not\subseteq B$ and $x_0x_1 \in E(B)$, Q contains a vertex x_α that is an attachment-vertex of B distinct from x_0. Since B has attachment-vertices only on T_i and T_j and Q is $\mathcal{T}$-clean, the vertex x_α is the only vertex of Q on T_j.

We claim that $\alpha \leqslant 2$. If B is trivial, then $\alpha = 1 \leqslant 2$, so suppose that B is non-trivial, and thus $B = B_k$ for some $k \in \{j+1, \dots, m\}$. We have $x_0 \in V(T_i)$ and $x_\alpha \in V(T_j)$, so by Claim 2, the path x_0Qx_α must intersect T_k^0 in a vertex $x_{\alpha'}$ with $0 < \alpha' < \alpha$. Since Q is $\mathcal{T}$-clean, the vertex $x_{\alpha'}$ is the only vertex of Q in $V(T_k)$. Hence, by definition of T_k, the vertices $x_{\alpha'-1}$ and $x_{\alpha'+1}$ are attachment-vertices of B_k, and therefore belong to $V(T_i) \cup V(T_j)$. The only vertex of Q in $V(T_i)$ is x_0, and the only vertex of Q in $V(T_j)$ is x_α, so $x_{\alpha'-1} = x_0$ and $x_{\alpha'+1} = x_\alpha$. Hence $\alpha' = 1$ and $\alpha = 2$. This proves $\alpha \leqslant 2$.

Since $Q \subseteq B_j - V(T_j^0)$, Claim 2 implies that the only component of F_{j-1} intersected by Q is T_i. Hence, $V(Q) \cap V(F_j) = \{x_0, x_\alpha\}$. Since $\alpha \leqslant 2$ and $p \geqslant 4$, the path $x_\alpha Q x_p$ is contained in a non-trivial F_j-bridge, which equals B_k for some $k \in \{j+1, \dots, m\}$. See Figure 4.4b. The F_j-bridge B is contained in B_j and has an attachment-vertex in A_j, and the vertex x_α is an attachment-vertex of B_k in $V(T_j)$. Hence, by Claim 8 applied to the trivial path consisting of the vertex x_α alone, we have $x_\alpha \in A_k^{\text{out}}$. By Claim 3, B_k is disjoint from T_j^0, and it is contained in the same component of $B_j - V(T_j^0)$ as Q. Hence, by Claim 2, B_k can only have attachment-vertices in $V(T_i)$ and $V(T_j)$. Since Q is $\mathcal{T}$-clean with $x_0 \in V(T_i)$ and $x_\alpha \in V(T_j)$, we have $x_{\alpha+1}Qx_p \subseteq B_k - A_k$. Since $x_\alpha \in A_k^{\text{out}}$, this implies $x_{\alpha+1} \in D_k \subseteq V(T_k^0)$, and thus $x_{\alpha+2} \in V(T_k)$. Therefore, $\{x_{\alpha+1}, x_{\alpha+2}\} \subseteq V(T_k)$, contrary to the assumption that Q is $\mathcal{T}$-clean. □

Although the partition $\mathcal{T}$ is 6-blocking, its parts can be arbitrarily large. The next step of our construction refines the chordal partition.



(a) $V(Q) \cap V(T_j^0) \neq \varnothing$ (b) $V(Q) \cap V(T_j^0) = \varnothing$

Figure 4.4. The two cases in the proof that the length of Q is at most 3.

5. Refinement of the Chordal Partition

In order to define our refinement of the chordal partition $\mathcal{T}$, we need to study its properties in more detail.

Claim 10. — *For each $j \in \{1, \dots, m\}$, $|D_j| < \Delta^{40}$.*

Proof. — If B_j has no attachment-vertices, then $|D_j| = 1 \leqslant \Delta^{40}$, so suppose that B_j has some attachment-vertices. B_j has attachment-vertices in at most two components of F_{j-1}, and on each of them A_j^{out} has one or two vertices, so $|A_j^{\text{out}}| \leqslant 4$. Since each vertex in D_j is at distance at most τ from a vertex in A_j^{out},

$$|D_j| \leqslant |A_j^{\text{out}}|(\Delta^0 + \ldots + \Delta^\tau) < 4\Delta^{\tau+1} < \Delta^{\tau+3} = \Delta^{40}. \qquad \square$$

Two paths in a graph are *internally disjoint* if none of them contains an inner vertex of another, and a path is *internally disjoint* from a set of vertices D if no inner vertex of the path belongs to D.

Claim 11. — *For each $j \in \{1, \dots, m\}$, the tree T_j^0 is the union of a family $\mathcal{P}_j$ of at most $2\Delta^{40}$ geodesic paths which are pairwise internally disjoint and internally disjoint from D_j.*

Proof. — Let S denote the set of all vertices of T_j^0 with degree at least 3. Since T_j^0 is the tree in $B_j - A_j$ which contains all vertices in D_j and has the smallest possible number of edges, every leaf of T_j^0 belongs to D_j, so T_j^0 has at most $|D_j|$ leaves, and thus $|S| \leqslant |D_j|$. The tree T_j^0 is a subdivision of a tree with vertex-set $S \cup D_j$. Therefore, T_j^0 is the union of a set $\mathcal{P}$ of at most $|S \cup D_j|$ pairwise internally disjoint paths such that each $P \in \mathcal{P}$

has its ends in $S \cup D_j$ and is internally disjoint from $S \cup D_j$. We have $|\mathcal{P}| \leqslant |S \cup D_j| \leqslant 2|D_j|$, so by Claim 10, $|\mathcal{P}| \leqslant 2\Delta^{40}$. Suppose towards a contradiction that some path $P \in \mathcal{P}$ is not geodesic in $B_j - A_j$, and let P' be a geodesic path in $B_j - A_j$ between the ends of P. Hence, P' has less edges than P, so any spanning tree of $P' \cup \bigcup_{Q \in \mathcal{P} \setminus \{P\}} Q$ has less edges than T and contains all vertices in D_j, which is a contradiction. □

An important property of geodesic paths is that the distances between vertices are preserved in them. We show that the tree T_j^0 'approximates' the distances between its vertices in $B_j - A_j$.

Claim 12. — *Let $j \in \{1, \dots, m\}$, and let $x, y \in V(T_j^0)$. Then*

$$\operatorname{dist}_{T_j^0}(x, y) < \Delta^{40} \operatorname{dist}_{B_j - A_j}(x, y).$$

Proof. — Let P be a geodesic x–y path in $B_j - A_j$. We have $\operatorname{dist}_P(x, y) = \operatorname{dist}_{B_j - A_j}(x, y)$, so we need to show that $\operatorname{dist}_{T_j^0}(x, y) < \Delta^{40} \operatorname{dist}_P(x, y)$.

First consider the case when P is internally disjoint from $V(T_j^0)$.

Let $z_0, \dots, z_s$ denote the sequence of all vertices of the path xT_j^0y that belong to $D_j \cup \{x, y\}$ or have degree at least 3 in T, ordered by increasing distance from x (so that $z_0 = x$ and $z_s = y$).

We show that $s < |D_j|$ by associating a distinct vertex $z_i' \in D_j$ to each z_i. Let $i \in \{0, \dots, s\}$. If $z_i \in D_j$, then let $z_i' := z_i$. Otherwise $z_i \notin D_j$, and either z_i is an end of xT_j^0y but not a leaf of T_j^0, or z_i has degree at least 3 in T_j^0. In both cases, there exists a leaf z_i' in T_j^0 such that z_i is adjacent to the component of $T_j^0 - V(xT_j^0y)$ that contains z_i'. By our choice of T_j^0, we have $z_i' \in D_j$. Clearly, the vertices $z_0', \dots, z_s'$ are distinct, so $s < |D_j|$.

For each $i \in \{0, \dots, s-1\}$, let $T(i)$ denote the graph obtained from T_j^0 by removing all edges and inner vertices of $z_iT_j^0z_{i+1}$ and adding the path P. The path P has ends in z_0 and z_s, and is internally disjoint from $V(T_j^0)$, so $T(i)$ is a tree. This tree still contains D, so by definition of T_j^0,

$$|E(T_j^0)| \leqslant |E(T(i))| = |E(T_j^0)| - |E(z_iT_j^0z_{i+1})| + |E(P)|,$$

so $\operatorname{dist}_{T_j^0}(z_i, z_{i+1}) = |E(z_iT_j^0z_{i+1})| \leqslant |E(P)|$. Therefore,

$$\begin{aligned}\operatorname{dist}_{T_j^0}(x, y) = \sum_{i=0}^{s-1} |E(z_iT_j^0z_{i+1})| \leqslant s \cdot |E(P)| < |D_j| \cdot \operatorname{dist}_P(x, y) \\ \leqslant \Delta^{40} \cdot \operatorname{dist}_P(x, y),\end{aligned}$$

where the last inequality follows from Claim 10.

It remains to consider the case when P has at least one inner vertex in $V(T_j^0)$. Let $w_0, \dots, w_n$ denote the vertices in $V(P) \cap V(T_j^0)$ ordered by increasing distance from x in P (so that $w_0 = x$ and $w_n = y$). For each $i \in \{0, \dots, n-1\}$, the path $w_i P w_{i+1}$ has no inner vertices in $V(T_j^0)$, so $\operatorname{dist}_{T_j^0}(w_i, w_{i+1}) < \Delta^{40} \cdot \operatorname{dist}_P(w_i, w_{i+1})$, and thus

$$\operatorname{dist}_{T_j^0}(x,y) \leqslant \sum_{i=0}^{n-1} \operatorname{dist}_{T_j^0}(w_i, w_{i+1}) < \sum_{i=0}^{n-1} \Delta^{40} \cdot \operatorname{dist}_P(w_i, w_{i+1}) = \Delta^{40} \cdot \operatorname{dist}_P(x,y). \qquad \square$$

Let $c \in \mathbb{N}$. For any vertex $x \in V(G)$, the number of vertices $x' \in V(G)$ with $\operatorname{dist}_G(x, x') \leqslant c$ is at most $\sum_{i=0}^{c} \Delta^i$, and therefore less than Δ^{c+1}. Therefore, for any edge $e \in E(G)$, the number of edges $e' \in E(G)$ with $\operatorname{dist}_G(e, e') \leqslant c$ is less than $2\Delta^{c+2}$ (since any such e' is incident to a vertex at distance at most c from one of the two endpoints of e). We use these bounds implicitly in the following part of this section.

In a graph J, we say that a set of edges $M \subseteq E(J)$ is *d-independent* if for any pair of distinct edges $e_1, e_2 \in M$ we have $\operatorname{dist}_J(e_1, e_2) > d$. We aim to refine the partition $\mathcal{T}$ be removing a set of edges $M_j \subseteq E(T_j^0)$ from each T_j, and letting the components of the resulting forests be the parts of the partition. The precise description of the desired properties of the sets M_j will be given in Claim 14. Roughly speaking, we want the edges in each M_j to be far away from each other, from other sets $M_{j'}$, and from the set D_j, while ensuring that the components of $T_j - M_j$ have bounded size. In order to formalise being far away, we need the following definition. Let $i \in \{1, \dots, m\}$, and suppose that the set $M_i \subseteq E(T_i^0)$ is already defined. Let S be a set of vertices or a set of edges in $B_i - V(T_i^0)$. The *mixed distance* of S from M_i is

$$\operatorname{mdist}_i(S) := \min\{\operatorname{dist}_{B_i - A_i}(M_i, v) + \operatorname{dist}_{B_i - V(T_i^0)}(v, S) : v \in V(T_i) \setminus V(T_i^0)\}.$$

Our goal is to construct the sets M_j so that for an appropriate constant c (specified in the next section), for each $j \in \{1, \dots, m\}$ with $X_j \neq \varnothing$, we have $\operatorname{mdist}_i(M_j) > c$ for all $i \in X_j$.

The sets M_j will be constructed one-by-one, where each set M_j is obtained from T_j^0 by selecting an appropriate set of edges from each geodesic path in $\mathcal{P}_j$, using the following claim, which exploits the fact that G is a plane graph.

Claim 13. — *Let $c \geqslant 1$, let $d := (8c+12)\Delta^{c+2}$, and let $n_0 \geqslant d + 2c$. Let P be a geodesic path in $B_j - A_j$ for some $j \in \{1, \dots, m\}$ with $X_j \neq \varnothing$, and suppose that for each $i \in X_j$ we are given a set $M_i \subseteq E(T_i^0)$ that is*

$(d+2c)$-independent in $B_i - A_i$, Then there exists a set $M_P \subseteq E(P)$ that is $(d+2c)$-independent in $B_j - A_j$ such that each component of $P - M_P$ has length at least $\min\{n_0, |E(P)|\}$ and less than $5n_0$, and for each $i \in X_j$ we have $\mathrm{mdist}_i(M_P) > c$.

Proof. — We may assume that the length of P is at least $5n_0$, as otherwise the lemma is satisfied by $M_P = \varnothing$. Let x and y denote the ends of P. Let $\{P_1, \dots, P_t\}$ be an inclusion-maximal family of pairwise vertex-disjoint subpaths of P each of length d such that $\mathrm{dist}_P(V(P_\alpha), V(P_\beta)) \geqslant n_0$ for distinct $\alpha, \beta \in \{1, \dots, t\}$, and $\mathrm{dist}_P(V(P_\alpha), \{x, y\}) \geqslant n_0$ for every $\alpha \in \{1, \dots, t\}$. Since $n_0 > d$ and the length of P is at least $5n_0$, we have $t > 0$. Consider any maximal subpath $P' \subseteq P$ internally disjoint from each of the paths $P_1, \dots, P_t$. Then the length of P' is at least n_0. Since our family of paths is inclusion-maximal, the length of P' is less than $d + 2n_0$ as otherwise we would be able to extend our family with a path of length d obtained from P' by removing at least n_0 vertices from each side. Since $d + 2n_0 < 3n_0$, we conclude that the length of any such P' is at least n_0 and less than $3n_0$.

We claim that each path P_α contains an edge e_α such that for every $i \in X_j$ we have $\mathrm{mdist}_i(e_\alpha) > c$. Since $|X_j| \leqslant 2$, it suffices to show that for each $i \in X_j$, there is less than $d/2$ edges $e \in E(P)$ with $\mathrm{mdist}_i(e) \leqslant c$. Fix $\alpha \in \{1, \dots, t\}$ and $i \in X_j$. Partition M_i into two sets M_i' and M_i'' by assigning each edge $e' \in M_i$ to M_i' if $\mathrm{dist}_{B_i - A_i}(e', V(P_\alpha)) \leqslant c$, and to M_i'' if $\mathrm{dist}_{B_i - A_i}(e', V(P_\alpha)) > c$. Since M_i is $(d+2c)$-independent in $B_i - A_i$, and the length of P_α is at most d, the set M_i' contains at most one edge.

For every $e' \in M_i''$, let $U_{e'}$ denote a subtree of G on all vertices at distance at most c from e' in $B_i - A_i$ such that each $u \in V(U_{e'})$ has the same distance from e' in $U_{e'}$ as in $B_i - A_i$ (one can think of $U_{e'}$ as a "BFS-spanning tree rooted at the edge e'") Since the set M_i is $(d+2c)$-independent in $B_i - A_i$, the trees $U_{e'}$ are pairwise vertex-disjoint, and by definition of M_i'' the trees $U_{e'}$ are disjoint from the path P_α. For each $e' \in M_i''$, let $Z_{e'} := N_{B_i}(V(U_{e'})) \cap A_i$. For each $z \in Z_{e'}$, define a $V(T_i)$–A_i path $Q(e', z)$ as follows. Let y be a vertex of $U_{e'}$ adjacent to z in B_i which minimises $\mathrm{dist}_{U_{e'}}(e', y)$ (and thus also minimises $\mathrm{dist}_{B_i - A_i}(e', y)$). Let x be the vertex on the path between y and e' which lies on $V(T_i)$ and minimises $\mathrm{dist}_{U_{e'}}(x, y)$ (this is well defined since e' has both ends in $V(T_i)$). Then the path $Q(e', z)$ is obtained from $xU_{e'}y$ by adding the vertex z attached to y. Each pair of distinct paths $Q_1 = Q(e_1', z_1)$ and $Q_2 = Q(e_2', z_2)$ is *consistent*, meaning that if their intersection $Q_1 \cap Q_2$ is not empty, then $Q_1 \cap Q_2$ is a path with an end in a common end of Q_1 and Q_2.

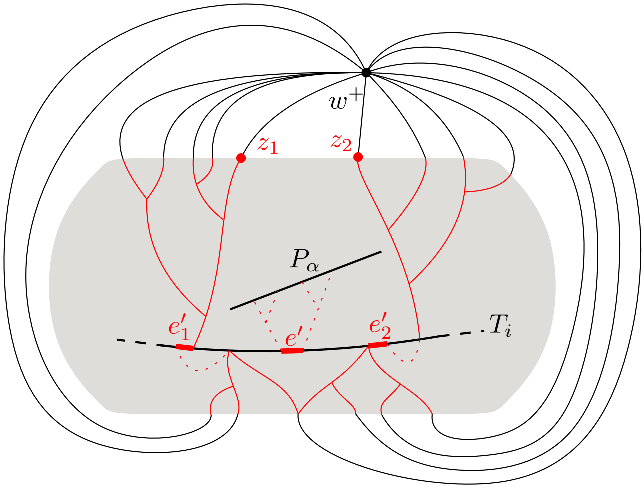


Figure 5.1. Illustration of a possible scenario in Claim 13. The set M'_i consists of the edge e', and the boundary of the face f contains two paths $Q(e'_1, z_1)$ and $Q(e'_2, z_2)$. Hence, $S = \{e'\} \cup E(Q(e'_1, z_1)) \cup E(Q(e'_2, z_2))$.

Let J denote the union of T_i and the paths $Q(e', z)$ for all $e' \in M''_i$ and $z \in Z_{e'}$. The graph J is a subgraph of B_i that intersects A_i only in the sets $Z_{e'}$. Thus, all vertices in the sets $Z_{e'}$ are incident with the outer face of B_i. Let J^+ denote the planar graph obtained from J by adding a new vertex w_+ on the outer-face of B_i and making it adjacent to all vertices in the sets $Z_{e'}$. See Figure 5.1. Since $P_\alpha \subseteq B_i - A_i$ and $\text{dist}_{B_i - A_i}(V(P_\alpha), M''_i) > c$, the path P_α is disjoint from J^+ and therefore the path P_α is contained in a face f of J^+. Let S denote the set of all edges in $E(J) \setminus E(T_i)$ on the boundary of f. Since the paths $Q(e', z)$ are pairwise consistent $V(T_i)$–A_i paths, the edges in S can be covered by the union of at most two paths of the form $Q(e', z)$. In particular, $|S| \leqslant 2c + 2$.

We claim that for every $e \in E(P_\alpha)$ with $\text{mdist}_i(e) \leqslant c$ we have $\text{dist}_{B_i}(S \cup M'_i, e) \leqslant c$. Suppose that $\text{mdist}_i(e) \leqslant c$. Hence, there exist $e' \in M_i$ and $v \in V(T_i) \setminus V(T_i^0)$ that satisfy $\text{dist}_{B_i - A_i}(e', v) + \text{dist}_{B_i - V(T_i^0)}(v, e) \leqslant c$. If $e' \in M'_i$, then we indeed have $\text{dist}_{B_i}(S \cup M'_i, e) \leqslant c$, so we assume that $e' \in M''_i$. Let R be a shortest path between v and e in $B_i - V(T_i^0)$. Since $\text{dist}_{B_i - A_i}(e, e') > c$, the path R must intersect A_i. Let z be the vertex of R that belongs to A_i and is closest to v on R. Hence, $z \in Z_{e'}$. Since J^+ contains $Q(e', z)$, the path $Q(e', z)$ is disjoint from the interior of f.

Therefore, the subpath of R between z and e must intersect the boundary of f in a vertex u, and since R is disjoint from T_i^0, the vertex u is an end of an edge in S. Therefore, $\operatorname{dist}_{B_i}(S,e) \leqslant c$. This completes the proof that for every $e \in E(P_\alpha)$, if $\operatorname{mdist}_i(e) \leqslant c$, then $\operatorname{dist}_{B_i}(S \cup M_i', e) \leqslant c$. Since $|S \cup M_i'| \leqslant |S| + |M_i'| \leqslant 2c+3$, for each $i \in X_j$ there exist less than $(4c+6)\Delta^{c+2}$ edges $e \in E(P_\alpha)$ with $\operatorname{mdist}_i(e) \leqslant c$. Since $|X_j| \leqslant 2$ and the length of P_α is $(8c+12)\Delta^{c+2}$, for each $\alpha \in \{1,\dots,t\}$ there exists an edge $e_\alpha \in E(P_\alpha)$ such that $\operatorname{mdist}_i(e_\alpha) > c$ for all $i \in X_j$.

Let $M_P := \{e_1,\dots,e_t\}$. Thus, $\operatorname{mdist}_i(M_P) > c$ for each $i \in X_j$. Since the distance between any two of the paths $P_1,\dots,P_t$ is at least n_0 on P, the set M_P is n_0-independent in P. Because P is geodesic in $B_j - A_j$ and $n_0 \geqslant d+2c$, the set M_P is $(d+2c)$-independent in $B_j - M_j$.

It remains to show that the components of $P - M_P$ have appropriate sizes. Let Q be a component of $P - M_P$. Since M_P contains one edge from each of the subpaths $P_1,\dots,P_t$, the path Q intersects at most two of the paths $P_1,\dots,P_t$, and the total number of edges of Q shared with $P_1,\dots,P_t$ is at most $2d$, and thus less than $2n_0$. The edges of Q that do not belong to any of the paths $P_1,\dots,P_t$ induce a maximal subpath of P internally disjoint from each of the paths $P_1,\dots,P_t$, which thus has length at least n_0 and less than $3n_0$. Hence, the length of P' is at least n_0 and less than $5n_0$. □

Finally, we are ready to construct the sets M_j.

Claim 14. — *Let $c \geqslant 1$ and $d := (8c+12)\Delta^{c+2}$. There exists a family $\{M_j \subseteq E(T_j^0) : j \in \{1,\dots,m\}\}$ such that for every $j \in \{1,\dots,m\}$:*

(a) M_j *is $(d+2c)$-independent in* $B_j - A_j$,
(b) $\operatorname{dist}_{T_j^0}(D_j, M_j) \geqslant 2\Delta^{40}$,
(c) *for each* $i \in X_j$, $\operatorname{mdist}_i(M_j) > c$,
(d) *each component of* $T_j^0 - M_j$ *has at most* $10\Delta^{80}(d+2c)$ *vertices, and*
(e) *for any pair of vertices* $x,y \in V(T_j^0)$ *satisfying* $\operatorname{dist}_{B_j-A_j}(x,y) \leqslant d+2c$ *and* $E(xT_j^0y) \cap M_j \neq \varnothing$, *we have* $\operatorname{dist}_{T_j^0}(x,y) = \operatorname{dist}_{B_j-A_j}(x,y)$.

Proof. — Let $n_0 := \Delta^{40}(d+2c)$.

We construct the sets M_j by induction on j. Let $j \in \{1,\dots,m\}$, and suppose that the sets M_i with $i < j$ have already been constructed. In particular, each M_i is $(d+2c)$-independent in $B_i - A_i$. We now construct M_j.

By Claim 11 there is a family $\mathcal{P}_j$ of at most $2\Delta^{40}$ pairwise internally disjoint geodesic paths in $B_j - A_j$ whose union is T_j^0, and which are internally disjoint from D_j. Observe that every inner vertex of a path $P \in \mathcal{P}_j$ has degree two in T_j^0.

For each $P \in \mathcal{P}_j$, let $M_P \subseteq E(P)$ be the subset of edges obtained by applying Claim 13 to c, n_0 and P. Thus, M_P is $(d+2c)$-independent in $B_j - A_j$, $\operatorname{mdist}_i(M_P) > c$ for each $i \in X_j$, and each component of $P - M_P$ is a path of length at least $\min\{n_0, |E(P)|\}$ and less than $5n_0$.

We show that the set $M_j := \bigcup\{M_P : P \in \mathcal{P}_j\}$ satisfies the claim. For the proof of (a), we need to show that M_j is $(d+2c)$-independent in $B_j - A_j$. Suppose towards a contradiction that there are distinct $e_1, e_2 \in M_j$ with $\operatorname{dist}_{B_j - A_j}(e_1, e_2) \leqslant d + 2c$. By Claim 12,

$$\operatorname{dist}_{T_j^0}(e_1, e_2) < \Delta^{40} \operatorname{dist}_{B_j - A_j}(e_1, e_2) \leqslant \Delta^{40}(d + 2c) = n_0.$$

However, if $P \in \mathcal{P}_j$ is the path containing e_1, then the shortest path between e_1 and e_2 in T_j^0 contains a component of $P - M_P$, and therefore has length at least $\min\{n_0, |E(P)|\} = n_0$ (since $e_1 \in E(P)$); that is, $\operatorname{dist}_{T_j^0}(e_1, e_2) \geqslant n_0$, a contradiction.

For any $P \in \mathcal{P}_j$ and $e \in M_P$, the distance between e and the ends of P is at least $n_0 = \Delta^{40}(d + 2c)$, and therefore at least $2\Delta^{40}$. Since the paths in $\mathcal{P}_j$ are pairwise internally disjoint, and internally disjoint from D_j, this implies that $\operatorname{dist}_{T_j^0}(D_j, M_j) \geqslant 2\Delta^{40}$. Therefore (b) is satisfied.

By definition of the sets M_P, for any $P \in \mathcal{P}_j$ and $i \in X_j$ we have $\operatorname{mdist}_i(M_P) > c$, and therefore for each $i \in X_j$ we have $\operatorname{mdist}_i(M_j) > c$. This proves (c).

For (d), observe that if a component T' of $T_j^0 - M_j$ intersects a path $P \in \mathcal{P}_j$, then $T' \cap P$ is a component of $P - M_P$, so it has less than $5n_0$ edges, and therefore at most $5n_0$ vertices. Hence,

$$|V(T')| \leqslant |\mathcal{P}_j| \cdot 5n_0 \leqslant 2\Delta^{40} \cdot 5\Delta^{40}(d + 2c) = 10\Delta^{80}(d + 2c).$$

Finally, for the proof of (e), let $x, y \in V(T_j^0)$ be vertices satisfying $\operatorname{dist}_{B_j - A_j}(x, y) \leqslant d + 2c$ and $E(xT_j^0y) \cap M_j \neq \varnothing$. Let $e \in E(xT_j^0y) \cap M_j$, and let $P \in \mathcal{P}_j$ be the path containing e. By Claim 12,

$$\operatorname{dist}_{T_j^0}(x, y) < \Delta^{40} \cdot \operatorname{dist}_{B_j - A_j}(x, y) \leqslant \Delta^{40} \cdot (d + 2c) = n_0.$$

Since $M_P \neq \varnothing$, every component of $P - M_P$ has length at least n_0, so the path xT_j^0y does not contain a component of $P - M_P$. Since $E(xT_j^0y) \cap M_P \neq \varnothing$ and all inner vertices of P have degree two in T_j^0, this implies that xT_j^0y is a subpath of P, and since P is geodesic in $B_j - A_j$, we have $\operatorname{dist}_{T_j^0}(x, y) = \operatorname{dist}_{B_j - A_j}(x, y)$. □

6. Analysis of the Partition

Let $\ell := 222$, and let $c := 2\ell+6 = 450$. Fix a family $\{M_j : j \in \{1,\dots,m\}\}$ satisfying Claim 14 for our value of c. Let $\mathcal{R}$ denote the partition of $V(G)$ where each part is the vertex-set of a component of $\bigcup_{j=1}^{m}(T_j - M_j)$. By Claim 14(d), for each $j \in \{1,\dots,m\}$, the size of every component of $T_j^0 - M_j$ is at most $10\Delta^{80}((8c+12)\Delta^{c+2}+2c) = 10\Delta^{80}(3612\,\Delta^{452}+900)$. Since each component of $T_j - M_j$ can be obtained from a component of $T_j^0 - M_j$ by attaching at most Δ vertices to each vertex of the component, the width of $\mathcal{R}$ is at most $(\Delta+1)\cdot 10\Delta^{80}(3612\,\Delta^{452}+900)$.

To complete the proof of Theorem 3.1, we show that $\mathcal{R}$ is ℓ-blocking; that is, no $\mathcal{R}$-clean path in G has length greater than ℓ. Since a subpath of an $\mathcal{R}$-clean path is $\mathcal{R}$-clean, it suffices to show that there is no $\mathcal{R}$-clean path of length exactly $\ell+1$, so in our analysis we focus only on paths of length at most $\ell+1$.

We start by proving some properties of $\mathcal{R}$-clean paths.

Claim 15. — *Let $j \in \{1,\dots,m\}$, and let $Q = (x_0,\dots,x_q)$ be an $\mathcal{R}$-clean path in $B_j - A_j$ with $\{x_0,x_q\} \subseteq V(T_j)$ and $q \in \{1,\dots,\ell+1\}$. Then $E(x_0T_jx_q) \cap M_j \neq \varnothing$, and for each $e \in E(x_0T_jx_q) \cap M_j$ we have $\operatorname{dist}_{T_j}(e,x_\alpha) \leqslant q+2$ for $\alpha \in \{0,q\}$. In particular,*

$$\operatorname{dist}_{T_j}(M_j,x_\alpha) \leqslant q+2 \quad \text{for } \alpha \in \{0,q\}.$$

Proof. — For each $\alpha \in \{0,q\}$, let x'_α denote the vertex x_α if $x_\alpha \in V(T_j^0)$, or the vertex in $V(T_j^0)$ that is adjacent to x_α in T_j if $x_\alpha \notin V(T_j^0)$. Hence, x'_α is in the same part of $\mathcal{R}$ as x_α and $\operatorname{dist}_{T_j}(x'_\alpha,x_\alpha) \leqslant 1$. In order to apply Claim 14(e) to x'_0 and x'_q, observe that

$$\begin{aligned}\operatorname{dist}_{B_j-A_j}(x'_0,x'_q) &\leqslant \operatorname{dist}_{T_j}(x'_0,x_0) + \operatorname{dist}_Q(x_0,x_q) + \operatorname{dist}_{T_j}(x_q,x'_q)\\ &\leqslant q+2\\ &\leqslant \ell+3\\ &< (8c+12)\Delta^{c+2}+2c.\end{aligned}$$

Furthermore, since Q is $\mathcal{R}$-clean, the part of $\mathcal{R}$ containing x_0 and x'_0 is distinct from the part containing x_q and x'_q, so $E(x'_0T_j^0x'_q) \cap M_j \neq \varnothing$. Therefore, by Claim 14(e),

$$\operatorname{dist}_{T_j^0}(x'_0,x'_q) = \operatorname{dist}_{B_j-A_j}(x'_0,x'_q) \leqslant q+2.$$

Since $M_j \subseteq E(T_j^0)$, we have $E(x_0T_jx_q) \cap M_j = E(x'_0T_j^0x'_q) \cap M_j \neq \varnothing$. Let $e \in E(x'_0T_j^0x'_q) \cap M_j$. The length of the path $x'_0T_j^0x'_q$ is at most $q+2$, so for each $\alpha \in \{0, q\}$ we have $\mathrm{dist}_{T_j^0}(e, x'_\alpha) \leqslant q + 1$, and therefore

$$\mathrm{dist}_{T_j}(e, x_\alpha) = \mathrm{dist}_{T_j}(e, x'_\alpha) + \mathrm{dist}_{T_j}(x'_\alpha, x_\alpha) \leqslant (q + 1) + 1 = q + 2. \quad \square$$

Claim 16. — *Let $j \in \{1, \ldots, m\}$, and let $Q = (x_0, \ldots, x_q)$ be an $\mathcal{R}$-clean path in B_j with $q \in \{0, \ldots, \ell + 1\}$ that is internally disjoint from A_j. Then $|V(Q) \cap V(T_j)| \leqslant 2$.*

Proof. — Suppose to the contrary that there exist distinct vertices $y_1, y_2, y_3 \in V(Q) \cap V(T_j)$. Since $T_j \subseteq B_j - A_j$ and Q is internally disjoint from A_j, each subpath of Q between two of the vertices y_1, y_2, y_3 is contained in $B_j - A_j$. By Claim 15 applied to $y_1 Q y_2$, there exists an edge $e \in E(y_1 T_j y_2) \cap M_j$ with $\mathrm{dist}_{B_j - A_j}(e, y_1) \leqslant q + 2$ and $\mathrm{dist}_{B_j - A_j}(e, y_2) \leqslant q + 2$. Without loss of generality, y_3 belongs to the same component of $T_j - e$ as y_1, and therefore $e \notin E(y_1 T_j y_3)$. By Claim 15 applied to $y_1 Q y_3$, there exists an edge $e' \in E(y_1 T_j y_3) \cap M_j$ with $\mathrm{dist}_{B_j - A_j}(e', y_1) \leqslant q + 2$. Therefore, $e \neq e'$, and

$$\begin{aligned}\mathrm{dist}_{B_j - A_j}(e, e') &\leqslant \mathrm{dist}_{B_j - A_j}(e, y_1) + \mathrm{dist}_{B_j - A_j}(y_1, e') \\ &\leqslant 2(q + 2) \leqslant 2\ell + 6 = c,\end{aligned}$$

which contradicts Claim 14(a). $\square$

Claim 17. — *Let $j \in \{1, \ldots, m\}$, and let $Q = (x_0, \ldots, x_q)$ be an $\mathcal{R}$-clean path in B_j with $q \in \{0, \ldots, \ell + 1\}$ that is internally disjoint from A_j, such that x_0 is an attachment-vertex of B_j on a tree T_i with $i \in X_j$ and $\mathrm{dist}_{B_i - A_i}(x_0, M_i) \leqslant \ell + 3$. Then $|V(Q) \cap V(T_j)| \leqslant 1$.*

Proof. — Suppose to the contrary that $|V(Q) \cap V(T_j)| > 1$. Hence, by Claim 16, we have $|V(Q) \cap V(T_j)| = 2$, say $V(Q) \cap V(T_j) = \{x_\alpha, x_\beta\}$ for some $\alpha, \beta \in \{0, \ldots, q\}$ with $\alpha < \beta$. Since $x_0 \in V(T_i)$, we have $\alpha \geqslant 1$. Since $\{x_\alpha, x_\beta\} \subseteq V(T_j) \subseteq B_j - A_j$ and Q is internally disjoint from A_j, we have $x_1 Q x_\beta \subseteq B_j - A_j$. By Claim 15 applied to $x_\alpha Q x_\beta$, we have

$$\mathrm{dist}_{B_j - A_j}(x_\alpha, M_j) \leqslant \beta - \alpha + 2 \leqslant q - \alpha + 2 \leqslant \ell - \alpha + 3.$$

The vertex x_0 is an attachment-vertex of B_j in $V(T_i)$, $i \in X_j$, and by Claim 5, we have $B_j \subseteq B_i$, and by Claim 3, B_j has no attachment-vertices in $V(T_i^0)$, so $B_j - A_j \subseteq B_j \subseteq B_i - V(T_i^0)$. Therefore,

$$\begin{aligned}\mathrm{mdist}_i(M_j) &\leqslant \mathrm{dist}_{B_i - A_i}(M_i, x_0) + \mathrm{dist}_{B_i - V(T_i^0)}(x_0, M_j) \\ &\leqslant (\ell + 3) + (\mathrm{dist}_{B_j}(x_0, x_\alpha) + \mathrm{dist}_{B_j - A_j}(x_\alpha, M_j)) \\ &\leqslant (\ell + 3) + \alpha + (\ell - \alpha + 3) = 2\ell + 6 = c.\end{aligned}$$

This contradicts Claim 14(c). $\square$

Next, we bound the length of $\mathcal{R}$-clean paths in some special cases. Recall that $F_j = \bigcup_{i<j} T_i$.

Claim 18. — *Let $i, j \in \{1, \dots, m\}$ with $i < j$, and let $Q = x_0 \cdots x_q$ be an $\mathcal{R}$-clean path with $q \in \{1, \dots, \ell + 1\}$, $x_0 \in V(T_i)$, $x_q \in V(T_j)$, and $V(Q) \cap V(F_j) = \{x_0, x_q\}$. Then*

(a) *if $\operatorname{dist}_{B_i - A_i}(x_0, M_i) \leqslant \ell + 3$ or $\operatorname{dist}_{B_j - A_j}(x_q, M_j) \leqslant \ell + 3$, then $q \leqslant 2$;*

(b) *otherwise, $q \leqslant 4$.*

Proof. — Since $V(Q) \cap V(F_j) = \{x_0, x_q\}$, the path Q is contained in some F_j-bridge. If that F_j-bridge is trivial, then $q = 1$ and the claim follows. Hence, Q is contained in a non-trivial F_j-bridge, which equals B_k for some $k > j$ by Claim 6. By Claim 2, the set $V(T_k^0)$ separates the vertices x_0 and x_q in B_k, so some inner vertex of Q must lie on T_k^0.

Let x_α be an inner vertex of Q in $V(T_k^0)$. Since the vertices $x_{\alpha-1}$ and $x_{\alpha+1}$ are adjacent to $V(T_k^0)$ in B_k, they belong to $A_k \cup V(T_k)$. Since the only attachment-vertices of B_k on Q are $x_0 \in V(T_i)$ and $x_q \in V(T_j)$, we conclude that $x_{\alpha-1} \in \{x_0\} \cup V(T_k)$ and $x_{\alpha+1} \in \{x_q\} \cup V(T_k)$. By Claim 16, we have $|V(Q) \cap V(T_k)| \leqslant 2$. Since $x_\alpha \in V(T_k^0) \subseteq V(T_k)$, at most one of the vertices $x_{\alpha-1}$ and $x_{\alpha+1}$ lies on T_k. In particular, $x_{\alpha-1} = x_0$ or $x_{\alpha+1} = x_q$, so $\alpha \in \{1, q-1\}$. If $x_{\alpha-1} = x_0$ and $x_{\alpha+1} = x_q$, then $\alpha = 1$ and $q = 2$, and the claim holds. Hence we may assume that one of $x_{\alpha-1}$ and $x_{\alpha+1}$ lies on T_k, and therefore $V(Q) \cap V(T_k) = \{x_1, x_2\}$ or $V(Q) \cap V(T_k) = \{x_{q-2}, x_{q-1}\}$. Thus, there exists $\beta \in \{1, q-2\}$ such that $V(Q) \cap V(T_k) = \{x_\beta, x_{\beta+1}\}$. By Claim 15 applied to the path $x_\beta Q x_{\beta+1}$, we have $\operatorname{dist}_{B_k - A_k}(x_\beta, M_k) \leqslant 3$ and $\operatorname{dist}_{B_k - A_k}(x_{\beta+1}, M_k) \leqslant 3$.

For the proof of (a), suppose that $\operatorname{dist}_{B_i - A_i}(x_0, M_i) \leqslant \ell + 3$ or $\operatorname{dist}_{B_j - A_j}(x_q, M_j) \leqslant \ell + 3$. Hence, by Claim 17, we have $|V(Q) \cap V(T_k)| = 1$, so $x_{\alpha-1} = x_0$ and $x_{\alpha+1} = x_q$, and therefore $q = 2$. This proves (a).

Next, we show (b). Suppose that $\beta = 1$. Then $x_q \in V(T_j)$, $x_2 \in V(T_k)$, $V(x_2 Q x_q) \cap V(F_k) = \{x_q, x_2\}$, and $\operatorname{dist}_{B_k - A_k}(x_2, M_k) \leqslant 3 < \ell + 3$. Hence, by (a), the length of $x_2 Q x_q$ is at most 2, and therefore $q \leqslant 4$. The case when $\beta = q - 2$ is similar: We have $x_0 \in V(T_i)$, $x_{q-2} \in V(T_k)$, $V(x_0 Q x_{q-2}) \cap V(F_k) = \{x_0, x_{q-2}\}$, and $\operatorname{dist}_{B_k - A_k}(x_{q-2}, (M_k)) \leqslant 3 < \ell + 3$. Hence, by (a), the length of $x_0 Q x_{q-2}$ is at most 2, so $q \leqslant 4$. This completes the proof of (b). □

Claim 19. — *Let $i \in \{1, \dots, m\}$, and let $Q = x_0 \cdots x_q$ be an $\mathcal{R}$-clean path with $q \in \{0, \dots, \ell + 1\}$ and $\{x_0, x_q\} \subseteq V(Q) \cap V(F_i) \subseteq V(T_i)$. Then $q \leqslant 4$.*

Proof. — If $q = 0$, then the claim holds trivially, so we assume that $x_0 \neq x_q$. By Claim 16, we have $|V(Q) \cap V(F_i)| \leqslant 2$, so $V(Q) \cap V(F_i) = \{x_0, x_q\}$. Since $(V(T_j) \colon j \in \{1, \dots, m\})$ is a partition of $V(G)$, each inner vertex of Q belongs to some tree T_j with $j > i$. Let T_j be the tree containing an inner vertex of Q with the smallest j. Thus, Q intersects F_{j-1} only in its ends, and B_j is the F_{j-1}-bridge containing Q. By Claim 5, $B_j \subseteq B_i$. Since B_j has attachment-vertices in $V(T_i)$, we have $i \in X_j$. By Claim 15, we have $\mathrm{dist}_{T_i}(x_0, M_i) \leqslant q + 2 \leqslant \ell + 3$ and $\mathrm{dist}_{T_i}(x_q, M_i) \leqslant q + 2 \leqslant \ell + 3$. Hence, by Claim 17, we have $|V(Q) \cap V(T_j)| = 1$, say $V(Q) \cap V(T_j) = \{x_\alpha\}$. By Claim 18(a), each of the paths $x_0 Q x_\alpha$ and $x_\alpha Q x_q$ has length at most 2, so $q \leqslant 4$. □

For each $j \in \{1, \dots, m\}$, the graph B_j intersects at most three components of F_j, namely, T_j and at most two components of F_{j-1} on which B_j has attachment-vertices. We aim to show that every $\mathcal{R}$-clean path in B_j with both ends on F_j has length at most 36 (the value $\tau = 37$ was chosen to be greater than this bound). We first prove the following helper claim.

Claim 20. — *Let $i, j \in \{1, \dots, m\}$ with $i < j$, let $Q = (x_0, \dots, x_q)$ be an $\mathcal{R}$-clean path in B_j with $q \in \{0, \dots, \ell + 1\}$, $V(Q) \cap V(T_i) = \{x_0\}$ and $x_q \in V(F_j)$. Then $q \leqslant 8(a - 1)$, where $a \in \{1, 2, 3\}$ is the number of components of F_j that intersect Q.*

Proof. — If $a = 1$, then since $V(Q) \cap V(T_i) = \{x_0\}$, the only component of F_j intersecting Q is T_i, and thus $x_q = x_0$, so $q = 0 = 8(a - 1)$. Hence, we assume that $a \geqslant 2$. Let $T_{i'}$ be the component of F_j that contains a vertex of Q, is distinct from T_i and has i' as small as possible. Let x_α and x_β denote respectively the first and the last vertex of Q in $V(T_{i'})$. By Claim 18, the length of $x_0 Q x_\alpha$ is at most 4, and by Claim 19, the length of $x_\alpha Q x_\beta$ is also at most 4. Hence, $\beta \leqslant 8$. If $i' = j$, then by our choice of i', we have $V(Q) \cap V(F_j) \subseteq \{x_0\} \cup V(T_j)$, so $x_q \in V(T_j)$, and therefore $q = \beta \leqslant 8 \leqslant 8(a-1)$. Hence, assume that $i' \neq j$ which means that $i' \in X_j$. The path Q intersects T_i and $T_{i'}$, and by Claim 2 it intersects also T_j, so $a = 3$. We have $V(x_\beta Q x_q) \cap V(T_{i'}) = \{x_\beta\}$, and the path $x_\beta Q x_q$ intersects at most two components of F_j, so we already know that its length is at most $8(2 - 1) = 8$, so $q \leqslant \beta + 8 \leqslant 16 = 8(a - 1)$. □

Claim 21. — *Let $j \in \{1, \dots, m\}$, and let $Q = (x_0, \dots, x_q)$ be an $\mathcal{R}$-clean path in B_j with $p \in \{0, \dots, \ell + 1\}$ such that $\{x_0, x_p\} \subseteq V(F_j)$. Then $q \leqslant 36$.*

Proof. — Let T_i be the tree intersecting Q with the smallest i. So $i \in X_j \cup \{j\}$. Let x_α and x_β denote the first and the last vertex of Q in $V(T_i)$.

By Claim 19, the length of $x_\alpha Q x_\beta$ is at most 4. If $i = j$, then $\alpha = 0$ and $\beta = q$, so $q \leqslant 4$. Therefore, we assume that $i < j$. We have $V(x_0 Q x_\alpha) \cap V(T_i) = \{x_\alpha\}$ and $V(x_\beta Q x_q) \cap V(T_i) = \{x_\beta\}$. Hence, by Claim 20, each of the paths $x_0 Q x_\alpha$ and $x_\beta Q x_q$ has length at most 16, which implies that $q \leqslant 16 + 4 + 16 = 36$. □

We proceed to the main part of the proof of Theorem 3.1. Towards a contradiction, assume that $\mathcal{R}$ is not ℓ-blocking. Hence, there exists an $\mathcal{R}$-clean path $P = (x_0, \dots, x_p)$ with $p > \ell$. Every subpath of an $\mathcal{R}$-clean path is $\mathcal{R}$-clean, so we may assume without loss of generality that the length of P is exactly $\ell+1$. Let T_i be the tree intersecting Q that has the smallest i. Let x_α and x_β denote respectively the first and the last vertex of Q belonging to T_i. By Claim 19, the length of $x_\alpha P x_\beta$ is at most 4. Hence, there exists $Q \in \{x_0 P x_\alpha, x_\beta P x_p\}$ with length at least $\lceil((\ell+1)-4)/2\rceil = 110$. The path Q intersect $V(F_i)$ only in one of its ends, and that end lies on T_i. Therefore, to reach a contradiction and complete the proof it suffices to show the following claim.

Claim 22. — *Let $i \in \{1, \dots, m\}$, and let $Q = (x_0, \dots, x_q)$ be an $\mathcal{R}$-clean path with $q \in \{0, \dots, \ell+1\}$ and $V(Q) \cap V(F_i) = \{x_0\} \subseteq V(T_i)$. Then $q \leqslant 109$.*

Claim 22 is a consequence of the following technical claim.

Claim 23. — *Let $i \in \{1, \dots, m\}$, let $Q = (x_0, \dots, x_q)$ be an $\mathcal{R}$-clean path contained in an F_i-bridge such that $q \in \{74, \dots, \ell+1\}$ and $V(Q) \cap V(T_i) = \{x_0\}$. Then*

(a) *there exist $i' \in \{1, \dots, m\}$ and an $\mathcal{R}$-clean path $Q' = (x'_0, \dots, x'_{q'})$ contained in an $F_{i'}$-bridge such that $q' \in \{q-36, \dots, \ell+1\}$, $V(Q') \cap V(T_{i'}) = \{x'_0\}$ and $\mathrm{mdist}_{i'}(x'_0) \leqslant 39$, and*

(b) *there exist $j \in \{i+1, \dots, m\}$ such that $i \in X_j$ and $\mathrm{dist}_{B_i - V(T_i^0)}(x_0, M_j) \leqslant 42$.*

Before proving Claim 23, we show how it implies Claim 22.

Proof of Claim 22 assuming Claim 23. — Towards a contradiction, suppose that $q \geqslant 110$. We will apply Claim 23(a) to (i, Q) to obtain a pair (i', Q'), and then we will apply Claim 23(b) to (i', Q') to obtain an index j' with contradictory properties.

We have $q \geqslant 110$, so in particular, $q \in \{74, \dots, \ell+1\}$. Since $V(Q) \cap V(F_i) = \{x_0\} \subseteq V(T_i)$, we have $V(Q) \cap V(T_i) = \{x_0\}$ and the path Q is contained in an F_i-bridge, so i and Q satisfy the preconditions of Claim 23. By Claim 23(a), there exist $i' \in \{1, \dots, m\}$ and an $\mathcal{R}$-clean path $Q' = (x'_0, \dots, x'_{q'})$ contained in an $F_{i'}$-bridge such that $q' \in \{q - 36, \dots, q\} \subseteq \{74, \dots, \ell+1\}$, $V(Q) \cap V(T_{i'}) = \{x'_0\}$ and $\mathrm{mdist}_{i'}(x'_0) \leqslant 39$. Hence, i' and Q' satisfy the preconditions of Claim 23. By Claim 23(b) applied to i' and Q', there exist $j' \in \{i'+1, \dots, m\}$ such that $i' \in X_{j'}$ and $\mathrm{dist}_{B_{i'} - V(T^0_{i'})}(x'_0, M_j) \leqslant 42$. Therefore,

$$\mathrm{mdist}_{i'}(M_{j'}) \leqslant \mathrm{mdist}_{i'}(x'_0) + \mathrm{dist}_{B_{i'} - V(T^0_{i'})}(x'_0, M_{j'}) \leqslant 39 + 42 = 81,$$

which contradicts Claim 14(c) since $c = 450 > 81$. ☐

The proof of Claim 23 makes use of the following claim:

Claim 24. — *Let $j \in \{1, \dots, m\}$, let $Q = (x_0, \dots, x_q)$ be an $\mathcal{R}$-clean path in $B_j - V(T^0_j)$ with $q \in \{0, \dots, \ell+1\}$ such that $x_0 \in A_j$, and there exists an F_j-bridge B contained in B_j with $x_0 \in V(B)$ and $x_q \notin V(B)$. Then $q \leqslant 37$.*

Proof. — Let x_α be the last vertex on Q that belongs to $V(F_j)$. By Claim 21, we have $\alpha \leqslant 36$. Unless $q = \alpha \leqslant 36$, the path $x_\alpha Q x_q$ is contained in a non-trivial F_k-bridge which equals B_k for some $k \in \{j+1, \dots, m\}$ by Claim 6. Let $x_{\alpha'}$ be the first vertex of Q that belongs to $V(B_k)$. By Claim 8, we have $x_{\alpha'} \in A^{\mathrm{out}}_k$. Towards a contradiction, suppose that $q \geqslant 38$, and thus $q \geqslant \alpha + 2$. Since $\mathrm{dist}_G(x_{\alpha'}, x_\alpha) \leqslant 36$, we have $x_{\alpha+1} \in D_k \subseteq V(T^0_k)$, and therefore $x_{\alpha+2} \in V(T_k)$. By definition of T_k, there exists a vertex $x'_{\alpha+2} \in V(T^0_k)$ that belongs to the same component of $T_k - M_k$ as $x_{\alpha+2}$ and satisfies $\mathrm{dist}_{T_k}(x_{\alpha+2}, x'_{\alpha+2}) \leqslant 1$. In particular, $\mathrm{dist}_{B_k - A_k}(x_{\alpha+1}, x'_{\alpha+2}) \leqslant 2$. By Claim 12, the length of the path $x_{\alpha+1} T^0_k x'_{\alpha+2}$ is less than $2\Delta^{40}$. Since Q is $\mathcal{R}$-clean, we have $E(x_{\alpha+1} T^0_k x'_{\alpha+2}) \cap M_k \neq \varnothing$, and therefore $\mathrm{dist}_{T^0_j}(D_k, M_k) < 2\Delta^{40}$, which contradicts Claim 14(b). ☐

It remains to prove Claim 23.

Proof of Claim 23. — Since $V(Q) \cap V(F_i) = \{x_0\}$ and $q \geqslant 74 > 1$, Q is contained in a non-trivial F_i-bridge, so by Claim 3, we have $x_0 \notin V(T^0_i)$, and by Claim 6, there exists $j \in \{i+1, \dots, m\}$ with $Q \subseteq B_j$. Fix the largest $j \in \{i+1, \dots, m\}$ with $Q \subseteq B_j$. We split the argument into two cases based on whether Q intersects T^0_j or not.

Case 1. $V(Q) \cap V(T^0_j) = \varnothing$. Let B be the F_j-bridge containing the edge x_0x_1. Hence, $B \subseteq B_j$, and B has attachment-vertices in $V(T_i)$ and $V(T_j)$. We have $x_q \in V(B)$ because otherwise Claim 24 would imply $q \leqslant 37$ contrary to our assumption that $q \geqslant 74$. Therefore $\{x_0, x_1, x_q\} \subseteq V(B)$, and in particular, B is non-trivial. By Claim 6, we have $B = B_k$ for some $k \in \{j+1, \dots, m\}$. By our choice of j, we have $Q \nsubseteq B_k$. Hence there exist

$\alpha, \beta \in \{0, \dots, q\}$ with $\alpha < \beta$ such that $\{x_\alpha, x_\beta\} \subseteq V(B_k)$, $x_\alpha Q x_\beta$ is edge-disjoint from B_k and $x_\beta Q x_q \subseteq B_k$. Since $x_0 x_1 \in E(B_k)$, we have $\alpha > 0$. The vertices x_α and x_β are attachment-vertices of B_k. Since x_0 is the only vertex of Q in $V(T_i)$, the vertices x_α and x_β lie on T_j. By Claim 19, the length of $x_\alpha Q x_\beta$ is at most 4, and we have $\mathrm{dist}_{B_j - A_j}(x_\beta, M_j) \leqslant 6$ by Claim 15. In particular, $\mathrm{mdist}_j(x_\beta) \leqslant 6$. By Claim 21, we have $\beta \leqslant 36$, so (a) is satisfied by $i' = j$ and $Q' = x_\beta Q x_q$. Furthermore, $\mathrm{dist}_{B_i - V(T_i^0)}(x_0, M_j) \leqslant \beta + 6 \leqslant 42$, so j satisfies (b).

Case 2. $V(Q) \cap V(T_j^0) \neq \varnothing$. Let x_α be the last vertex of Q in $V(T_j^0)$. By Claim 21, we have $\alpha \leqslant 36$. Since $x_\alpha \in V(T_j^0)$, we have $x_{\alpha+1} \in A_j \cup V(T_j)$. Suppose towards a contradiction, that $x_{\alpha+1} \in A_j$. By Claim 21, we have $\alpha+1 \leqslant 36$. Then $x_\alpha x_{\alpha+1}$ is a trivial F_j-bridge contained in B_j that contains $x_{\alpha+1}$ and does not contain x_q. By Claim 24 applied to $x_{\alpha+1} Q x_q$, the length of $x_{\alpha+1} Q x_q$ is at most 37, so $q \leqslant (\alpha+1)+37 \leqslant 36+37 < 74$, a contradiction. Therefore, $x_{\alpha+1} \notin A_j$, so $x_{\alpha+1} \in V(T_j)$. By Claim 15 applied to $x_\alpha Q x_{\alpha+1}$, we have $\mathrm{dist}_{B_j - A_j}(x_{\alpha+1}, M_j) \leqslant 3$, and thus $\mathrm{dist}_{B_i - V(T_i^0)}(x_0, M_j) \leqslant (\alpha + 1) + 3 \leqslant 36 + 3 = 39$. This proves (b).

For the proof of (a), let x_β and x_γ denote the last two vertices of Q in $V(F_j)$ where $\beta < \gamma$. Since $\{x_\alpha, x_{\alpha+1}\} \subseteq V(T_j)$, we have $\beta \geqslant \alpha$, and by Claim 21, we have $\gamma \leqslant 36$. We have $\{x_\beta, x_\gamma\} \subseteq V(B_j) \cap V(F_j) = A_j \cup V(T_j)$. We consider three subcases.

Subcase 2.1. $x_\gamma \in V(T_j)$. By definition of x_α, the path $x_{\alpha+1} Q x_\gamma$ is disjoint from T_j^0, so $x_{\alpha+1} \in V(T_j) \setminus V(T_j^0)$ and (a) is satisfied by $i' = j$ and $Q' = x_\gamma Q x_q$ since

$$\begin{aligned}\mathrm{mdist}_j(x_\gamma) &\leqslant \mathrm{dist}_{B_j - A_j}(M_j, x_{\alpha+1}) + \mathrm{dist}_{B_j - V(T_j^0)}(x_{\alpha+1}, x_\gamma)\\ &\leqslant \alpha + 1 + 3\\ &\leqslant 36 + 3 = 39.\end{aligned}$$

Subcase 2.2. $x_\gamma \in A_j$ and $x_\beta \in V(T_j)$. The path $x_\beta Q x_\gamma$ is internally disjoint from F_j, so it is contained in an F_j-bridge B such that $B \subseteq B_j$. We have $x_q \in V(B)$, since otherwise by Claim 24 applied to $x_\gamma Q x_q$ the length of $x_\gamma Q x_q$ is at most 37, so $q \leqslant \gamma + 37 \leqslant 36 + 37 < 74$, which is a contradiction. Hence, $x_q \in V(B)$. Therefore, B is non-trivial, and by Claim 3, we have $B \subseteq B_j - V(T_j^0)$. Since $\gamma < q$, x_q is not an attachment-vertex of B, and we have $x_\gamma Q x_q \subseteq B$, so $x_\beta Q x_q \subseteq B \subseteq B_j - V(T_j^0)$. Thus,

(a) is satisfied by $i' = j$ and $Q' = x_\beta Q x_q$ since

$$\begin{aligned}\operatorname{mdist}_j(x_\beta) &\leqslant \operatorname{dist}_{B_j - A_j}(M_j, x_{\alpha+1}) + \operatorname{dist}_{B_j - V(T_j^0)}(x_{\alpha+1}, x_\beta)\\ &\leqslant \alpha + 1 + 3\\ &\leqslant 36 + 3 = 39.\end{aligned}$$

Subcase 2.3 $x_\gamma \in A_j$ and $x_\beta \in A_j$.

By the invariant from the construction of the forests F_j, the F_{j-1}-bridge B_j has attachments on exactly one tree $T_{i'}$ with $i' \neq i$, and the vertices x_γ and x_β lie on that tree $T_{i'}$ because the only vertex of Q on T_i is x_0. By Claim 19, the length of $x_\beta Q x_\gamma$ is at most 4, and by Claim 15, we have $\operatorname{dist}_{B_{i'} - A_{i'}}(M_{i'}, x_\gamma) \leqslant 6$. In particular, $\operatorname{mdist}_{i'}(x_\gamma) \leqslant 6$. Hence, (a) is satisfied by i' and $Q' = x_\gamma Q x_q$. This completes the proof. □

7. Graphs on Surfaces

This section proves Theorem 3.2 which lifts our result for blocking partitions of planar graphs (Theorem 3.1) to graphs on surfaces. We need the following folklore lemma (implicit in [17, 9] for example).

Lemma 7.1. — *For every connected graph G with Euler genus g and for every* BFS*-layering $(V_0, V_1, \dots)$ of G, G contains a tree T that is the union of at most $2g$ vertical paths with respect to $(V_0, V_1, \dots)$ such that $G - V(T)$ is planar.*

The next lemma is stated in terms of the following subgraph variant of clean paths: Let G be a graph and $\mathcal{Z}$ be a connected partition of a subgraph Z of G. A path P in G is *$\mathcal{Z}$-clean* if $|V(P) \cap V| \leqslant 1$ for each $V \in \mathcal{Z}$.

Lemma 7.2. — *For any integers $g \geqslant 0$ and $\ell \geqslant 1$, every connected graph G with Euler genus g has a connected subgraph Z such that $G - V(Z)$ is planar and Z has a connected partition $\mathcal{Z}$ with width at most $2g((5g+1)\ell + 3)$ such that every $\mathcal{Z}$-clean path of length at most ℓ in G intersects at most three parts in $\mathcal{Z}$.*

Proof. — The $g = 0$ case holds trivially with Z the empty graph and $\mathcal{Z}$ the empty set. Now assume that $g \geqslant 1$. Let $(V_0, V_1, \dots)$ be a BFS-layering of G where $V_0 = \{r\}$ for some $r \in V(G)$. By Lemma 7.1, G contains a tree T that is the union of at most $2g$ vertical paths such that $G' := G - V(T)$ is planar. For $a, b \in \mathbb{N}_0$ where $a \leqslant b$, let $V_{[a,b]} := \bigcup(V_j : j \in \{a, \dots, b\})$ and $T_{[a,b]} := T[V(T) \cap V_{[a,b]}]$. For $i \in \mathbb{N}_0$, we inductively construct a sequence of tuples $(x_i, X_i, Z_i, \mathcal{X}_i, \mathcal{Z}_i)$ with the following properties:

(1) $x_0 = 0$ and $x_i \in \{x_{i-1} + 3g\ell + 1, \dots, x_{i-1} + 5g\ell + 1\}$ for all $i \geqslant 1$;
(2) X_i is an induced subgraph of G with $V(T_{[x_{i-1}+1,x_i]}) \subseteq V(X_i) \subseteq V_{[x_{i-2}+\ell+1,x_i]}$;
(3) Z_i is an induced subgraph of G with $V(T_{[0,x_i]}) \subseteq V(Z_i) \subseteq V_{[0,x_i]}$;
(4) $X_i = Z_i - V(Z_{i-1})$;
(5) $\mathcal{X}_i$ is a connected partition of X_i of width at most $2g((5g+1)\ell+3)$;
(6) $\mathcal{Z}_i$ is a connected partition of Z_i of width at most $2g((5g+1)\ell+3)$ where $\mathcal{Z}_i = \mathcal{Z}_{i-1} \cup \mathcal{X}_i$;
(7) Every path in $G - V(Z_{i-1})$ of length at most ℓ intersects at most one part in $\mathcal{X}_i$; and
(8) Every $\mathcal{Z}_i$-clean path in G of length at most ℓ intersects at most three parts in $\mathcal{Z}_i$.

Note that when $i := |V(G)|$, we have $x_i \geqslant |V(G)|$ and $T \subseteq Z_i$, which implies that $(Z_i, \mathcal{Z}_i)$ satisfies the lemma statement.

For $i = 0$, such a tuple exists with $X_i = G[\{r\}]$, $Z_i = X_i$, $\mathcal{X}_i = (\{r\})$, and $\mathcal{Z}_i = \mathcal{X}_i$. Now assume that $i \geqslant 1$ and such a tuple exists for $i-1$.

Let $x_{i,1} := x_{i-1} + 3g\ell + 1$ and $X_{i,1} := T_{[x_{i-1}+1,x_{i,1}]}$. Then $X_{i,1}$ is the union of at most $2g$ vertical paths, and thus has at most $2g$ components. Suppose $G - V(Z_{i-1})$ contains a path P of length at most ℓ that intersects at least two components of $X_{i,1}$. Let $x_{i,2} := \max\{\{j : V(P) \cap V_j \neq \varnothing\} \cup x_{i,1}\}$ and $X_{i,2} := G[V(T_{[x_{i-1}+1,x_{i,2}]}) \cup V(P)]$. Then $X_{i,2}$ has at most $2g-1$ components. Moreover, since P has length at most ℓ, it follows that $x_{i,2} \in \{x_{i,1}, \dots, x_{i,1} + \ell\}$ and $V(P) \subseteq V_{[x_{i-1}-\ell+1,x_{i,1}]}$, so $V(X_{i,2}) \subseteq V_{[x_{i-1}-\ell+1,x_{i,1}]}$. Iterate the above procedure until there is no path of length at most ℓ that intersects two components of $X_{i,j}$. Such a process must terminate within at most $2g$ iterations, since no path can exist if $X_{i,j}$ has only one component. As such, there exists $j \in \{1, \dots, 2g\}$ such that $x_{i,j} \in \{x_{i-1} + 3g\ell + 1, \dots, x_{i-1} + 5g\ell + 1\}$, $V(X_{i,j}) \subseteq V_{[x_{i-1}-2g\ell+1,x_{i,1}]} \subseteq V_{[x_{i-2}+\ell+1,x_{i,1}]}$ and every path in $G - V(Z_{i-1})$ of length at most ℓ intersects at most one component of $X_{i,j}$. Set $x_i := x_{i,j}$, $X_i := G[V(X_{i,j})]$, and $Z_i := G[Z_{i-1} \cup X_i]$. Let $\mathcal{X}_i$ be the connected partition of X_i where each part induces a component of X_i and $\mathcal{Z}_i := \mathcal{Z}_{i-1} \cup X_{i,j}$. We now show that the construction satisfies the desired properties.

By construction, (1), (2), (3), (4) and (7) hold clearly. For (5), since X_i is the union of at most $2g$ vertical paths of length at most $5g\ell+1$ together with the union of at most $2g$ paths of length at most ℓ, it follows that $|V(X_i)| \leqslant 2g((5g+1)\ell+3)$. Thus (5) holds and so, by induction, (6) holds. It remains to show (8). Let P be a $\mathcal{Z}_i$-clean path in G of length at most ℓ. If $V(P) \subseteq V_{[0,x_{i-2}+\ell]}$, then the claim follows by induction. So assume that $V(P) \cap V_{[x_{i-2}+\ell+1,x_i]} \neq \varnothing$. Since $V(Z_{i-2}) \subseteq V_{[0,x_{i-2}]}$, this implies $V(P) \cap V(Z_{i-2}) = \varnothing$. Thus P only intersects parts in $\mathcal{X}_{i-1} \cup \mathcal{X}_i$. As P has length at most ℓ, it follows by (7) that P only intersects at most one part of $\mathcal{X}_{i-1}$. Let $W := V(P) \cap V(Z_{i-1})$. Since P is $\mathcal{Z}_i$-clean, it follows that $|W| \leqslant 1$. So $P - W$ consists of at most two components that are $\mathcal{Z}_i$-clean paths in $G - V(Z_{i-1})$. By (7), each component of $P - W$ intersects at most one part in $\mathcal{X}_i$. So P intersects at most three parts in $\mathcal{Z}_i$, as required. □

Proof of Theorem 3.2. — Without loss of generality, we may assume that G is connected. By Lemma 7.2 with $\ell = 895$, G contains a subgraph Z such that $G' := G - V(Z)$ is planar and Z has a connected partition $\mathcal{Z}$ with width at most $8950g^2 + 1796g$ such that every path of length at most 895 in G intersects at most three parts in $\mathcal{Z}$. By Theorem 3.1, G' has a 222-blocking partition $\mathcal{R}'$ with width at most $10\Delta^{80}(3612\,\Delta^{452} + 900)$. Let $\mathcal{R} := \mathcal{R}' \cup \mathcal{Z}$, which is a connected partition of G with width at most $\max\{10\Delta^{80}(3612\,\Delta^{452} + 900), 8950g^2 + 1796g\}$. We claim that $\mathcal{R}$ is 894-blocking. Consider an $\mathcal{R}$-clean path P in G. Then P intersects at most three parts in $\mathcal{Z}$. Let $W := V(P) \cap V(Z)$. Since P is $\mathcal{R}$-clean, $|W| \leqslant 3$. Therefore, $P - W$ has at most four components, each of which is an $\mathcal{R}'$-clean path in G'. Since each $\mathcal{R}'$-clean path in G' has length at most 222, it follows that P has length at most $4 \cdot 222 + 6 = 894$. Hence $\mathcal{R}$ is 894-blocking. □

8. Reflections on Blocking Partitions

This section considers which graph classes have ℓ-blocking partitions of width at most c for some constants ℓ, c. Bounded maximum degree is necessary, even for trees.

Proposition 8.1. — *If every tree with maximum degree Δ has an ℓ-blocking partition of width at most c, then $c \geqslant \Delta$.*

Proof. — Let T be the complete $(\Delta - 1)$-ary rooted tree of height $\ell + 1$. So T has maximum degree Δ and every root-to-leaf path has length $\ell + 1$. Let $\mathcal{R}$ be an ℓ-blocking partition of T of width at most c. For the sake of contradiction, suppose $c < \Delta$. Then every non-leaf vertex of T has a child that belongs to a different part in $\mathcal{R}$. So T contains a root-to-leaf path P where every pair of consecutive vertices belong to different parts in $\mathcal{R}$. Moreover, no two non-consecutive vertices in P belong to the same part, since each part in $\mathcal{R}$ is connected. Hence P is an $\mathcal{R}$-clean path of length $\ell + 1$, which is a contradiction. □

On the other hand, bounded maximum degree is not enough.

Proposition 8.2. — *There are no constants $c, \ell \in \mathbb{N}$ such that every 4-regular graph has an ℓ-blocking partition of width at most c.*

Proof. — Suppose for the sake of contradiction that every 4-regular graph has an ℓ-blocking partition of width at most c. Erdős and Sachs [20] showed that for any integers $\Delta, g \geqslant 3$ there is a Δ-regular graph with girth at least g. Let G be a 4-regular n-vertex graph with girth $g \geqslant c + \ell + 2$. Consider an ℓ-blocking partition $\mathcal{R}$ of G with width at most c. Say that an edge $uv \in E(G)$ is *red* if $u, v \in V$ for some $V \in \mathcal{R}$, otherwise it is *blue*. Since $g > c$, each part $V \in \mathcal{R}$ induces a tree, so the total number of red edges is less than n. Thus the number of blue edges is more than $|E(G)| - n = n$. Hence there is a cycle C in G that consists of blue edges, which has length at least $g \geqslant \ell + 2$. Therefore C contains a path P of length $\ell + 1$ that consists of blue edges. If distinct vertices v, w in P are in the same part in $\mathcal{R}$, then G contains a cycle of length at most $c + \ell$, which contradicts the choice of g. Hence P is $\mathcal{R}$-clean, which is a contradiction. □

Proposition 8.2 says that for a graph class to admit bounded blocking partitions, some structural assumption in addition to bounded degree is necessary. Theorem 3.1 shows that bounded Euler genus is such an assumption. We now show that bounded treewidth is another such assumption.

Theorem 8.3. — *Every graph G has a 2-blocking partition with width at most*

$$1350(\mathrm{tw}(G) + 1)(\Delta(G))^2.$$

The proof of Theorem 8.3 relies on a new lemma concerning tree-partitions. We say that a rooted T-partition $(B_x \colon x \in V(T))$ of a graph G is *detached* if for every non-root node $y \in V(T)$ with parent $x \in V(T)$, each vertex in B_y is adjacent to at most one component of $G[B_x]$.

Lemma 8.4. — *Every graph G has a detached T-partition of width at most $90(\mathrm{tw}(G) + 1)\Delta(G)$, for some tree T with $\Delta(T) \leqslant 15\Delta(G)$*

The proof of Lemma 8.4 builds on a clever argument due to a referee of a paper by Ding and Oporowski [8] showing that graphs with bounded treewidth and bounded maximum degree have tree-partitions of bounded width (see also [40, 10]); see Appendix A for the details.

Proof of Theorem 8.3. — By Lemma 8.4, G has a detached T-partition $(B_x : x \in V(T))$ with width at most $90(\mathrm{tw}(G) + 1)\Delta(G)$ for some tree T with $\Delta(T) \leqslant 15\Delta(G)$ and root $z \in V(G)$. Let $(V_0, V_1, \dots)$ be a BFS-layering of G where $V_0 = \{z\}$. We say a part B_x is in level i if $x \in V_i$. Colour the edges of G as follows: each edge with two ends in one part B_x is coloured

red, and each edge with one end in a part B_x at level i and one end in a part B_y at level $i+1$ is coloured red if i is odd and blue if i is even. Let $\mathcal{R}$ be the connected partition of G where each part is the vertex-set of a component of the spanning subgraph of G consisting of the red edges. Observe that the vertex-sets of the components of $G[B_z]$ are in $\mathcal{R}$. Moreover, for every other part $X \in \mathcal{R}$, there is a node $x \in V(T)$ with children $y_1, \dots, y_{\deg_T(x)-1} \in V(T)$ such that $X \subseteq B_x \cup B_{y_1} \cup \dots \cup B_{y_{\deg_T(x)-1}}$. Since every node in $V(T) \setminus \{z\}$ has at most $15\Delta(G) - 1$ children, it follows that each part in $\mathcal{R}$ has at most $(15\Delta(G)) \cdot (90(\mathrm{tw}(G)+1)\Delta(G)) \leqslant 1350(\mathrm{tw}(G)+1)(\Delta(G))^2$ vertices.

For the sake of contradiction, suppose G contains an $\mathcal{R}$-clean path P of length at least 3. Since P is $\mathcal{R}$-clean, its edges are blue and so all edges of P are between levels i and $i+1$ for some even i, and all the vertices of P at level i belong to one part B_x. Since each edge of P is blue, the vertices of P alternate between vertices in B_x and vertices that belong to parts that are indexed by the children of x. Since P has length at least 3, P has an internal vertex w that belongs to B_y for some child y of z. Since P is $\mathcal{R}$-clean, w is adjacent to at least two components of $G[B_x]$, contradicting $(B_x : x \in V(T))$ being a detached tree-partition. So every $\mathcal{R}$-clean path in G has length at most 2, as required. □

9. Open Problems

We conclude with some open problems.

OPEN PROBLEM 1. — *What is the minimum integer ℓ for which there exists a function f such that every planar graph G has an ℓ-blocking partition with width at most $f(\Delta(G))$? We have proved that $\ell \leqslant 222$, although we have chosen to simplify our proof rather than optimise the constant.*

OPEN PROBLEM 2. — *Can Theorem 1.5 be proved with f bounded by a polynomial function of d, r, s? Our proof gives $f(d,r,s) \leqslant (sd)^{O(r!)}$.*

Consider the following open problems for k-planar graphs.

OPEN PROBLEM 3. — *What is the minimum integer c such that there is a function f for which every k-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ where $\mathrm{tw}(H) \leqslant c$? We know that $3 \leqslant c \leqslant 15\,288\,899$.*

OPEN PROBLEM 4. — *Is there a constant c and a polynomial function f such that every k-planar graph G is contained in $H \boxtimes P \boxtimes K_{f(k)}$ where $\mathrm{tw}(H) \leqslant c$? Our proof gives $f(k) \leqslant 2^{O(\lfloor k/2 \rfloor !)}$.*

Questions analogous to Open Problems 3 and 4 can be asked for other natural classes.

Finally, consider what other graph classes have an ℓ-blocking partitions?

Open Problem 5. — *Does there exist integers* $\ell, c \geqslant 1$ *such that every graph with maximum degree at most* 3 *has an* ℓ*-blocking partition of width at most* c*?*

Open Problem 6. — *For every* $t \in \mathbb{N}$*, does there exist* $k_t \in \mathbb{N}$ *and a function* f_t *such that every* K_t*-minor-free graph* G *has a* k_t*-blocking partition with width at most* $f_t(\Delta(G))$*?*

Acknowledgements

Research of M.D. supported by an Australian Government Research Training Program Scholarship. Research of R.H. completed at Monash University, where supported by an Australian Government Research Training Program Scholarship. Research of M.T.S. completed at Université Libre de Bruxelles, where supported by a PDR grant from the Belgian National Fund for Scientific Research (FNRS). Research of D.W. supported by the Australian Research Council.

A preliminary version of this paper appeared in the *Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications* (EUROCOMB'23), doi:10.5817/CZ.MUNI.EUROCOMB23-049.

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Appendix A. Detached Tree-Partitions

This appendix is devoted to the proof of Lemma 8.4. Recall that a rooted tree-partition $(B_x : x \in V(T))$ of a graph G is *detached* if for every non-root node $y \in V(T)$ with parent $x \in V(T)$, each vertex in B_y is adjacent to at most one component of $G[B_x]$.

Lemma A.1. — *For any graph G, for any non-empty set $S \subseteq V(G)$, there exists a set X such that:*

- $S \subseteq X \subseteq V(G)$;
- $|X| \leqslant 2|S| - 1$; *and*
- *each vertex in $G - X$ is adjacent to at most one component of $G[X]$.*

Proof. — Consider the following algorithm: Initialise $i := 0$ and $S_0 := S$. While there is a vertex v in $G - S_i$ adjacent to at least two components of $G[S_i]$, let $S_{i+1} := S_i \cup \{v\}$ and $i := i + 1$.

Say this algorithm stops at $i = m$. Let $X := S_m$. Then each vertex in $G-X$ is adjacent to at most one component of $G[X]$. Let c_j be the number of components of $G[S_j]$. By construction, $|S_j| = |S| + j$ and $c_j \leqslant c_0 - j \leqslant |S| - j$ for each $j \in \{0, \dots, m\}$. In particular, if $m \geqslant |S|-1$ then $c_{|S|-1} = 1$. Thus $m \leqslant |S| - 1$ and $|X| \leqslant |S| + m \leqslant 2|S| - 1$. □

The following lemma is the core of the proof of Lemma 8.4.

Lemma A.2. — *For $k, d \in \mathbb{N}$, for any graph G with $\operatorname{tw}(G) \leqslant k - 1$ and $\Delta(G) \leqslant d$, for any set $S \subseteq V(G)$ with $5k \leqslant |S| \leqslant 30kd$, there exists a detached tree-partition $(B_x : x \in V(T))$ of G with root $z \in V(T)$ such that:*

- $\Delta(T) \leqslant 15d$;
- $|B_x| \leqslant 90kd$ *for each* $x \in V(T)$;
- $S \subseteq B_z$;
- $|B_z| \leqslant 3|S| - 5k$; *and*
- $\deg_T(z) \leqslant \frac{|S|}{2k} - 1$.

Proof. — We proceed by induction on $|V(G)|$.

Case 1. $|V(G - S)| \leqslant 90kd$: Let T be the tree with $V(T) = \{y, z\}$ and $E(T) = \{yz\}$. Note that $\Delta(T) = 1 \leqslant 15d$ and $\deg_T(z) = 1 \leqslant \frac{|S|}{2k} - 1$. By Lemma A.1, there exists a set $B_z \subseteq V(G)$ such that $S \subseteq B_z$, $|B_z| \leqslant 2|S| - 1 \leqslant 3|S| - 5k \leqslant 90kd$ and every vertex in $V(G) - B_z$ is adjacent to at most two components of $G[B_z]$. Set $B_y := V(G) - B_z$. Then $|B_y| \leqslant |V(G) - S| \leqslant 90kd$ and every vertex in B_y is adjacent to at most one component of $G[B_z]$. Hence $(B_x : x \in V(T))$ is the desired detached tree-partition of G.

Now assume that $|V(G-S)| \geqslant 90kd$.

Case 2. $5k \leqslant |S| \leqslant 15k$: By Lemma A.1, there exists a set $B_z \subseteq V(G)$ such that $S \subseteq B_z$, $|B_z| \leqslant 2|S| \leqslant \min\{3|S|-5k, 30k\}$ and every vertex in $V(G)-B_z$ is adjacent to at most one component of $G[B_z]$. Let $S' := \bigcup\{N_G(v) \setminus B_z : v \in B_z\}$. So $|S'| \leqslant d|B_z| \leqslant 30kd$. If $|S'| < 5k$ then add $5k - |S'|$ vertices from $V(G-B_z-S')$ to S', so that $|S'| = 5k$. This is well-defined since $|V(G-B_z)| \geqslant 90kd - 30k \geqslant 5k$, implying $|V(G-B_z-S')| \geqslant 5k - |S'|$. By induction, there exists a detached tree-partition $(B_x : x \in V(T'))$ of $G - B_z$ with root $z' \in V(T')$ such that:

- $|B_x| \leqslant 90kd$ for each $x \in V(T')$;
- $\Delta(T') \leqslant 15d$;
- $S' \subseteq B_{z'}$;
- $|B_{z'}| \leqslant 3|S'| - 5k \leqslant 90kd$; and
- $\deg_{T'}(z') \leqslant \frac{|S'|}{2k} - 1 \leqslant 15d - 1$.

Let T be the rooted tree obtained from T' by adding a new root z adjacent to z'. So $(B_x : x \in V(T))$ is a tree-partition of G with width at most $\max\{90kd, |B_z|\} \leqslant \max\{90kd, 30k\} = 90kd$. By construction, $\deg_T(z) = 1 \leqslant \frac{|S|}{2k} - 1$ and $\deg_T(z') = \deg_{T'}(z') + 1 \leqslant (15d-1)+1 = 15d$. Every other vertex in T has the same degree as in T'. Hence $\Delta(T) \leqslant 15d$, as desired. Finally, since $(B_x : x \in V(T'))$ is detached and every vertex in $V(G) - B_z$ is adjacent to at most one component of $G[B_z]$, it follows that $(B_x : x \in V(T))$ is also detached.

Case 3. $15k \leqslant |S| \leqslant 30kd$: By the separator lemma of Robertson and Seymour [38, (2.6)], there are induced subgraphs G_1 and G_2 of G with $G_1 \cup G_2 = G$ and $|V(G_1 \cap G_2)| \leqslant k$, where $|S \cap V(G_i)| \leqslant \frac{2}{3}|S|$ for each $i \in \{1,2\}$. Let $S_i := (S \cap V(G_i)) \cup V(G_1 \cap G_2)$ for each $i \in \{1,2\}$.

We now bound $|S_i|$. For a lower bound, since $|S \cap V(G_1)| \leqslant \frac{2}{3}|S|$, we have $|S_2| \geqslant |S \setminus V(G_1)| \geqslant \frac{1}{3}|S| \geqslant 5k$. By symmetry, $|S_1| \geqslant 5k$. For an upper bound, $|S_i| \leqslant \frac{2}{3}|S| + k \leqslant 20kd + k \leqslant 30kd$. Also note that $|S_1| + |S_2| \leqslant |S| + 2k$.

We have shown that $5k \leqslant |S_i| \leqslant 30kd$ for each $i \in \{1,2\}$. Thus we may apply induction to G_i with S_i the specified set. Hence there exists a detached tree-partition $(B^i_x : x \in V(T_i))$ of G_i with root $z_i \in V(T_i)$ such that:

- $|B^i_x| \leqslant 90kd$ for each $x \in V(T_i))$;
- $\Delta(T_i) \leqslant 15d$;
- $S_i \subseteq B_{z_i}$;
- $|B_{z_i}| \leqslant 3|S_i| - 5k$; and
- $\deg_{T_i}(z_i) \leqslant \frac{|S_i|}{2k} - 1$.

Let T be the rooted tree obtained from the disjoint union of T_1 and T_2 by identifying z_1 and z_2 into a new root vertex z. Let $B_z := B^1_{z_1} \cup B^2_{z_2}$. Let $B_x := B^i_x$ for each $x \in V(T_i) \setminus \{z_i\}$. Since $G = G_1 \cup G_2$ and $V(G_1 \cap G_2) \subseteq B^1_{z_1} \cap B^2_{z_2} \subseteq B_z$, we have that $(B_x : x \in V(T))$ is a tree-partition of G. By construction, $S \subseteq B_z$ and since $V(G_1 \cap G_2) \subseteq B^i_{z_i}$ for each i,

$$\begin{aligned}
|B_z| &\leqslant |B^1_{z_1}| + |B^2_{z_2}| - |V(G_1 \cap G_2)| \\
&\leqslant (3|S_1| - 5k) + (3|S_2| - 5k) - |V(G_1 \cap G_2)| \\
&= 3(|S_1| + |S_2|) - 10k - |V(G_1 \cap G_2)| \\
&\leqslant 3(|S| + 2|V(G_1 \cap G_2)|) - 10k - |V(G_1 \cap G_2)| \\
&\leqslant 3|S| + 5|V(G_1 \cap G_2)| - 10k \\
&\leqslant 3|S| - 5k \\
&< 90kd.
\end{aligned}$$

Every other part has the same size as in the tree-partition of G_1 or G_2. So this tree-partition of G has width at most $90kd$. Note that

$$\begin{aligned}
\deg_T(z) = \deg_{T_1}(z_1) + \deg_{T_2}(z_2) &\leqslant (\tfrac{|S_1|}{2k} - 1) + (\tfrac{|S_2|}{2k} - 1) \\
&= \tfrac{|S_1|+|S_2|}{2k} - 2 \\
&\leqslant \tfrac{|S|+2k}{2k} - 2 \\
&= \tfrac{|S|}{2k} - 1 \\
&< 15d.
\end{aligned}$$

Every other node of T has the same degree as in T_1 or T_2. Thus $\Delta(T) \leqslant 15d$. So it remains to show that $(B_x\colon x \in V(T))$ is detached. By induction, it follows that for every node $x \in V(T) \setminus \{z\}$ with child y, every vertex in B_y is adjacent to at most one component of $G[B_x]$. Now suppose that a vertex $v \in V(G) - B_z$ is adjacent to at least two components of $G[B_z]$. Let $u, w \in B_z$ be neighbours of v in G that belong to distinct components of $G[B_z]$. Since $(B^i_x\colon x \in V(T_i))$ is a detached tree-partition of G_i, it follows that either $u \in V(G_1) \setminus V(G_2)$ and $w \in V(G_2) \setminus V(G_1)$, or $u \in V(G_2) \setminus V(G_1)$ and $w \in V(G_1) \setminus V(G_2)$. As such, $v \in V(G_1) \cap V(G_2)$, but this is a contradiction since $V(G_1) \cap V(G_2) \subseteq B_z$. So $(B_x\colon x \in V(T))$ is detached, which completes the proof. □

Proof of Lemma 8.4. — First suppose that $|V(G)| < 5(\text{tw}(G) + 1)$. Let T be the 1-vertex tree with $V(T) = \{x\}$, and let $B_x := V(G)$. Then $(B_x : x \in V(T))$ is the desired detached tree-partition, since $|B_x| = |V(G)| < 5(\text{tw}(G) + 1) \leqslant 90(\text{tw}(G) + 1)\Delta(G)$ and $\Delta(T) = 0 \leqslant 15\Delta(G)$. Now assume that $|V(G)| \geqslant 5(\mathrm{tw}(G)+1)$. The result follows from Lemma A.2, where S is any set of $5(\mathrm{tw}(G)+1)$ vertices in G. $\square$

Manuscript received 21st September 2023,
accepted 3rd September 2024.

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Innovations in Graph Theory is a member of the
Mersenne Center for Open Publishing
ISSN: 3050-743X