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INNOVATIONS IN GRAPH THEORY

DIAMETER OF THE INVERSION GRAPH

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Abstract. — In an oriented graph $\vec{G}$, the inversion of a subset X of vertices consists in reversing the orientation of all arcs with both endvertices in X. The inversion graph of a labelled graph G, denoted by $\mathcal{I}(G)$, is the graph whose vertices are the labelled orientations of G in which two labelled orientations $\vec{G}_1$ and $\vec{G}_2$ of G are adjacent if and only if there is a set X whose inversion transforms $\vec{G}_1$ into $\vec{G}_2$. In this paper, we study the inversion diameter of a graph which is the diameter of its inversion graph denoted by $\mathrm{diam}(\mathcal{I}(G))$. We show that the inversion diameter is tied to the star chromatic number, the acyclic chromatic number and the oriented chromatic number. Thus a graph class has bounded inversion diameter if and only if it also has bounded star chromatic number, acyclic chromatic number and oriented chromatic number. We give some upper bounds on the inversion diameter of a graph G contained in one of the following graph classes: planar graphs ($\mathrm{diam}(\mathcal{I}(G)) \leqslant 12$), planar graphs of girth at least 8 ($\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$), graphs with maximum degree Δ ($\mathrm{diam}(\mathcal{I}(G)) \leqslant 2\Delta - 1$), and graphs with treewidth at most t ($\mathrm{diam}(\mathcal{I}(G)) \leqslant 2t$).

We also show that determining the inversion diameter of a given graph is NP-hard.

1. Introduction

The concept of inversion was defined by Belkhechine et al. in [4] as follows. In an oriented graph $\vec{G}$, if X is a set of vertices of $\vec{G}$, the *inversion* of X consists in reversing the orientation of all arcs with both endvertices in X. In particular, the reversal of a single arc is an inversion. Belkhechine et al. in [4] studied the *inversion number*, denoted by $\mathrm{inv}(\vec{G})$, which is the minimum number of inversions that transform $\vec{G}$ into a directed acyclic graph. In particular, they proved that for every fixed k, determining whether a given tournament T has inversion number at most k is polynomial-time

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solvable. In contrast, Bang-Jensen et al. [3] proved that deciding whether a given oriented graph has inversion number 1 is NP-complete. The maximum $\mathrm{inv}(n)$ of the inversion numbers of all oriented graphs of order n has also been investigated. Independently, Aubian et al. [2] and Alon et al. [1] proved $n-2\sqrt{n\log n} \leqslant \mathrm{inv}(n) \leqslant n-\lceil \log(n+1)\rceil$. Recently Duron et al. [7] studied the minimum number of inversions to make a digraph k-arc-strong or k-strong.

Making an oriented graph acyclic or k-strong through inversions can be viewed as a special case of the reconfiguration problem, which, given an orientation of a graph, asks for the minimum number of inversions required to transform it into an orientation belonging to a specified set of orientations of the same graph. Each reconfiguration problem (see, for example, the survey by Nishimura [14]) can be represented by a reconfiguration graph, whose vertices correspond to the set of configurations, with an edge between two configurations whenever one can be obtained from the other via a single transformation. In the case of inversions, this graph is defined as follows. Let G be a graph with vertices labelled $v_1,\dots,v_n$. The *(labelled) inversion graph* of G, denoted by $\mathcal{I}(G)$, is the graph whose vertices are the labelled orientations of G in which two labelled orientations $\vec{G}_1$ and $\vec{G}_2$ of G are adjacent if and only if there is a set X whose inversion transforms $\vec{G}_1$ into $\vec{G}_2$.

The first question in studying a reconfiguration problem is whether its reconfiguration graph is connected; in other words, whether any configuration can be transformed into any other via a sequence of allowed transformations. In the case of inversions, the inversion graph of any graph is trivially connected, since any orientation can be transformed into any other by reversing arcs one at a time. Once connectivity is established, the next natural question concerns the diameter of the reconfiguration graph, as it provides an upper bound on the minimum number of transformations required for any reconfiguration.

In this paper, we study the *(labelled) inversion diameter* of a graph G, which is the diameter of $\mathcal{I}(G)$, denoted by $\mathrm{diam}(\mathcal{I}(G))$. Making an oriented graph acyclic clearly needs at most as many inversions as transforming it into a specific acyclic oriented graph. Hence the inversion diameter of a graph is at least as large as the inversion number of any of its orientations. Similarly, the inversion diameter is at least as large as the number of inversions required to transform any orientation of the graph into an orientation with some connectivity property, if such an orientation exists. We prove various bounds on $\mathrm{diam}(\mathcal{I}(G))$ depending on the structure of G.

We first determine the maximum inversion diameter over all graphs on n vertices.

THEOREM 1.1. — *Let G be a graph on n vertices. Then*

$$\operatorname{diam}(\mathcal{I}(G)) \leqslant n - 1.$$

This bound is tight for complete graphs (Theorem 2.5).

For sparser graphs, one can expect much better bounds. Actually, we characterise classes of graphs $\mathcal{G}$ for which there exists a constant C (depending only on $\mathcal{G}$) such that the inversion diameter of every graph in $\mathcal{G}$ is at most C. Indeed, we prove that the inversion diameter is tied[(1)] to some well-studied parameters, namely the star chromatic number χ_s, the acyclic chromatic number χ_a, and the oriented chromatic number χ_o, which we now define.

A *star-forest* is a forest whose components have diameter at most 2. The *star chromatic number* $\chi_s(G)$ (resp. *acyclic chromatic number* $\chi_a(G)$) of a graph G is the least integer k such that G has a proper k-vertex-colouring in which any union of two colour classes induces a star-forest (resp. a forest).

The *oriented chromatic number* $\chi_o(D)$ of an oriented graph D is the minimum integer k for which there exists a proper colouring $\phi\colon V(D) \to [k]$ such that for all $i, j \in [k]$, the arcs linking $\phi^{-1}(i)$ and $\phi^{-1}(j)$ either are all oriented from $\phi^{-1}(i)$ to $\phi^{-1}(j)$ or are all oriented from $\phi^{-1}(j)$ to $\phi^{-1}(i)$. Equivalently, $\chi_o(D)$ is the least integer k such that there exists a homomorphism from D to a tournament on k vertices. The *oriented chromatic number* $\chi_o(G)$ of a graph G is the maximum of $\chi_o(\vec{G})$ over all orientations $\vec{G}$ of G. The star chromatic number, acyclic chromatic number, and oriented chromatic number are known to be tied [9, 12, 15], and we add to this list of equivalent parameters the inversion diameter.

THEOREM 1.2. — *Let $\mathcal{G}$ be a class of graphs.*
The following are equivalent.

- *There is a constant C such that $\operatorname{diam}(\mathcal{I}(G)) \leqslant C$ for every $G \in \mathcal{G}$.*
- *There is a constant C_s such that $\chi_s(G) \leqslant C_s$ for every $G \in \mathcal{G}$.*
- *There is a constant C_a such that $\chi_a(G) \leqslant C_a$ for every $G \in \mathcal{G}$.*
- *There is a constant C_o such that $\chi_o(G) \leqslant C_o$ for every $G \in \mathcal{G}$.*

As a consequence, graphs of bounded maximum degree, graphs of bounded treewidth, and planar graphs, have all bounded inversion diameter. Since these theoretical bounds are not very accurate, we show more precise upper bounds for these specific classes. In particular, we show:

(1) Two graph parameters γ_1 and γ_2 are *tied* if there are two functions f_1 and f_2 such that $\gamma_1(G) \leqslant f_1(\gamma_2(G))$ and $\gamma_2(G) \leqslant f_2(\gamma_1(G))$ for every graph G.

(a) $\mathrm{diam}(\mathcal{I}(G)) \leqslant 12$ for every planar graph G (Corollary 7.5), and there are planar graphs with inversion diameter at least 5 (Proposition 7.6);
(b) $\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$ for every planar graph G of girth at least 8 (Theorem 7.7), and there are planar graphs of arbitrary large girth with inversion diameter 3 (Proposition 7.8);
(c) $\mathrm{diam}(\mathcal{I}(G)) \leqslant 2\Delta - 1$ for every graph G with maximum degree at most $\Delta \geqslant 1$ (Theorem 5.1), and $\mathrm{diam}(\mathcal{I}(G)) = \Delta(G)$ for every complete graph G (Theorem 2.5);
(d) $\mathrm{diam}(\mathcal{I}(G)) \leqslant 2t$ for every graph G of treewidth at most t (Theorem 6.2), and for every $t \geqslant 2$, there are graphs of treewidth t with inversion diameter $t + 2$ (Theorem 6.5).

We also show that determining the inversion diameter of a given graph is NP-hard, and that determining whether two given orientations of a given bipartite planar graph are at distance at most 2 in its inversion graph is NP-complete (Corollary 8.5).

Structure of the paper

In Section 2, we give first results on the inversion diameter. In Section 3, we introduce our main tool to prove upper bounds on the inversion diameter. In Section 4, we show that the inversion diameter is tied to the star chromatic number, the acyclic chromatic number, and the oriented chromatic number. In Section 5, we show that $\mathrm{diam}(\mathcal{I}(G)) \leqslant 2\Delta(G) - 1$ for every graph G with at least one edge, and we improve this bound for $\Delta(G) \leqslant 3$. In Section 7, we show various bounds for the maximum inversion diameter of graphs in some minor-closed classes. In Section 8, we show that computing distances in $\mathcal{I}(G)$ is hard. More precisely, for every $k \geqslant 2$, it is NP-hard to determine if two given orientations are at distance at most k, and it is NP-hard to determine if $\mathrm{diam}(\mathcal{I}(G)) \leqslant k$.

Notation

For every positive integer k, we denote by $[k]$ the set $\{1, \dots, k\}$. We denote by $\log(\cdot)$ the logarithm in base 2. Given a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Graphs do not contain loops or multiple edges, otherwise, we speak of *multigraphs*. Similarly, given an oriented graph $\vec{G}$, we denote by $V(\vec{G})$ its vertex set and by $A(\vec{G})$ its arc set.

If X is a set of vertices in G (resp. $\vec{G}$), we denote by $G\langle X\rangle$ (resp. $\vec{G}\langle X\rangle$) the subgraph (resp. oriented subgraph) induced by X.

We denote by $\delta(G)$ the minimum degree of G and by $\delta^*(G)$ the *degeneracy* of G, which is $\max\{\delta(H) \mid H \text{ subgraph of } G\}$. Equivalently, $\delta^*(G) \leqslant d$ if and only if there is an ordering $(v_1, \dots, v_n)$ of $V(G)$ such that for every $i \in [n]$, v_i has at most d neighbours in $\{v_j \mid j < i\}$. We also denote by $\mathrm{Ad}(G)$ the average degree of G, that is $\frac{2|E(G)|}{|V(G)|}$, and by $\mathrm{Mad}(G)$ the maximum of $\mathrm{Ad}(H)$ over all subgraphs H of G with at least one vertex. Note that $\mathrm{Mad}(G)/2 \leqslant \delta^*(G) \leqslant \mathrm{Mad}(G)$.

More generally, we use standard notation in graph theory, and we refer the reader to [5] for undefined notion or notation, such as the parameters δ, Δ, χ.

Now consider an oriented graph $\vec{G}$ and some $X \subseteq V(\vec{G})$. We denote by $\mathrm{Inv}(\vec{G}; X)$ the result of the inversion of X in $\vec{G}$, and more generally, if $X_1, \dots, X_t$ are sets of vertices in $\vec{G}$, $\mathrm{Inv}(\vec{G}; (X_i)_{i\in[t]})$ is the oriented graph resulting from the inversion of $X_1, X_2, \dots, X_t$. Note that $\mathrm{Inv}(\vec{G}; (X_i)_{i\in[t]})$ does not depend on the ordering of $X_1, \dots, X_t$, since an arc uv is reversed if and only if $|\{i \in [t] \mid u, v \in X_i\}|$ is odd.

We will often use arguments from linear algebra. The vectors and matrices considered will always be over the field $\mathbb{F}_2$ with two elements 0 and 1. Given a matrix M, we denote by $\mathrm{rk}(M)$ its rank over $\mathbb{F}_2$. We denote by $\mathbf{0}$ the vector in $\mathbb{F}_2^d$ constant to 0, for d a positive integer that will be clear from the context. Moreover, given two vectors $\mathbf{x} = (x_1, \dots, x_d), \mathbf{y} = (y_1, \dots, y_d)$ of the same dimension d, we denote by $\mathbf{x} \cdot \mathbf{y}$ their scalar product, that is $\sum_{i=1}^{d} x_i y_i$.

2. First results on the inversion diameter

In this section, we introduce some tools to compute $\mathrm{diam}(\mathcal{I}(G))$, and we apply them on some highly structured graphs.

Let $\vec{G}_1$ and $\vec{G}_2$ be two orientations of a graph G. If an edge e has the same orientation in $\vec{G}_1$ and $\vec{G}_2$, we say that $\vec{G}_1$ and $\vec{G}_2$ *agree* on e; otherwise we say that they *disagree* on e. We denote by $E_=$ the set of edges of G on which $\vec{G}_1$ and $\vec{G}_2$ agree and by $E_\neq$ the set of edges of G on which $\vec{G}_1$ and $\vec{G}_2$ disagree.

The following observation will be used implicitly throughout the paper.

Observation 2.1. — *Let G be a graph.*

(1) If H is a subgraph of G, then $\mathrm{diam}(\mathcal{I}(H)) \leqslant \mathrm{diam}(\mathcal{I}(G))$.

(2) If G has connected components $C_1, \dots, C_k$, then

$$\operatorname{diam}(\mathcal{I}(G)) = \max\{\operatorname{diam}(\mathcal{I}(C_1)), \dots, \operatorname{diam}(\mathcal{I}(C_k))\}.$$

Another important observation is the following lower bound on the inversion diameter.

Observation 2.2. — *If G has n vertices and m edges, then*

$$\operatorname{diam}(\mathcal{I}(G)) \geqslant \frac{m}{n}.$$

Proof. — This follows from the following double counting of the number of orientations of G. On the one hand, G has 2^m orientations, because each edge has two possible orientations ; on the other hand, starting from a fixed orientation of G, one can reach at most $(2^n)^t$ orientations of G when using at most t inversions. For $t = \operatorname{diam}(\mathcal{I}(G))$, this gives $2^{n \operatorname{diam}(\mathcal{I}(G))} \geqslant 2^m$, and so $\operatorname{diam}(\mathcal{I}(G)) \geqslant \frac{m}{n}$. □

Theorem 1.1 follows directly from the following lemma and an immediate induction.

Lemma 2.3. — *Let G be a graph and v a vertex in G. Then*

$$\operatorname{diam}(\mathcal{I}(G)) \leqslant \operatorname{diam}(\mathcal{I}(G - v)) + 1.$$

Proof. — Let $\vec{G}_1$ and $\vec{G}_2$ be two orientations of G. Let X' be the set of neighbours w of v such that $\vec{G}_1$ and $\vec{G}_2$ disagree on vw. Set $X = X' \cup \{v\}$. Note that v has the same out-neighbours and the same in-neighbours in $\operatorname{Inv}(\vec{G}_1; X)$ and $\vec{G}_2$. Now let $\mathcal{X}$ be a collection of $\operatorname{diam}(\mathcal{I}(G-v))$ subsets of $V(G-v)$ such that $\operatorname{Inv}(\operatorname{Inv}(\vec{G}_1; X) - v; \mathcal{X}) = \vec{G}_2 - v$. We obtain $\operatorname{Inv}(\vec{G}_1; \mathcal{X} \cup \{X\}) = \vec{G}_2$ and hence the statement follows. □

Now we give a fundamental observation that allows us to use tools from linear algebra to bound $\operatorname{diam}(\mathcal{I}(G))$.

Observation 2.4. — *For every graph G and every positive integer t, the following are equivalent.*

(1) $\operatorname{diam}(\mathcal{I}(G)) \leqslant t$.

(2) For every function $\pi\colon E(G) \to \mathbb{F}_2$, there is a family $(\mathbf{u})_{u \in V(G)}$ of vectors in $\mathbb{F}_2^t$ such that

$$\pi(uv) = \mathbf{u} \cdot \mathbf{v}$$

for every edge $uv \in E(G)$.

Proof. — First suppose that $\operatorname{diam}(\mathcal{I}(G)) \leqslant t$, and let $\pi\colon E(G) \to \mathbb{F}_2$ be a function. Let $\vec{G}_1, \vec{G}_2$ be two orientations of G such that, for every edge $uv \in E(G)$, $\vec{G}_1$ and $\vec{G}_2$ agree on uv if and only if $\pi(uv) = 0$. By assumption,

there are sets $X_1, \dots, X_t \subseteq V(G)$ such that $\vec{G}_1 = \mathrm{Inv}(\vec{G}_2; X_1, \dots, X_t)$. For every vertex $u \in V(G)$, we define $\mathbf{u}$ to be the vector in $\mathbb{F}_2^t$ whose i^{th} coordinate is 1 if and only if $u \in X_i$, for every $i \in [t]$. Now observe that for every edge $uv \in E(G)$, uv is reversed by the inversions of $X_1, \dots, X_t$ if and only if $\mathbf{u} \cdot \mathbf{v} = 1$. Hence $\mathbf{u} \cdot \mathbf{v} = \pi(uv)$.

Reciprocally, given two orientations $\vec{G}_1$ and $\vec{G}_2$ of G, for every edge $uv \in E(G)$, let $\pi(uv) = 0$ if and only if $\vec{G}_1$ and $\vec{G}_2$ agree on uv. By assumption, there is a family $(\mathbf{u})_{u \in V(G)}$ of vectors in $\mathbb{F}_2^t$ such that $\pi(uv) = \mathbf{u} \cdot \mathbf{v}$ for every edge $uv \in E(G)$. We define $X_i = \{u \in V(G) \mid \mathbf{u}_i = 1\}$ for every $i \in [t]$. Then for every edge $uv \in E(G)$, uv is reversed by the inversions of $X_1, \dots, X_t$ if and only if $\mathbf{u} \cdot \mathbf{v} = 1$, and so $\vec{G}_2 = \mathrm{Inv}(\vec{G}_1; X_1, \dots, X_t)$. This proves that $\mathrm{diam}(\mathcal{I}(G)) \leqslant t$. □

We now fully determine the inversion diameter of balanced complete multipartite graphs. The purpose of this result is two-fold. Firstly, its proof may serve as a warm-up exercise for the following sections. Secondly, it gives in particular the inversion diameter of complete graphs, which serve as tightness examples in several results in the following sections. We denote by $K_r[\overline{K_t}]$ the complete r-partite graph in which every part has size t.

THEOREM 2.5. — *For all positive integers r and t, we have*

$$\mathrm{diam}\left(\mathcal{I}(K_r[\overline{K_t}])\right) = (r-1)t.$$

Proof. — Let the graph G be defined by $V(G) = [r] \times [t]$, and $E(G) = \{(i,j)(i',j') \mid i \neq i'\}$. Observe that G is isomorphic to $K_r[\overline{K_t}]$. By repeatedly applying Lemma 2.3 for all vertices in $V(G) \setminus (\{1\} \times [t])$ and using the fact that the inversion diameter of edgeless graphs is 0, we have $\mathrm{diam}(\mathcal{I}(G)) \leqslant (r-1)t + \mathrm{diam}(\mathcal{I}(\overline{K_t})) = (r-1)t$. Let us now prove that $\mathrm{diam}(\mathcal{I}(G)) \geqslant (r-1)t$.

Consider the function $\pi \colon E(G) \to \mathbb{F}_2$ defined by $\pi((i,j)(i',j')) = 1$ if and only if $|i'-i| = 1$ and $j = j'$, for all $i, i' \in [r]$ and $j, j' \in [t]$ with $i \neq i'$. Suppose that there is a family $(\mathbf{v}_{i,j})_{i \in [r], j \in [t]}$ of vectors in $\mathbb{F}_2^k$ such that $\pi((i,j)(i',j')) = \mathbf{v}_{i,j} \cdot \mathbf{v}_{i',j'}$ for every $i, i' \in [r]$ and $j, j' \in [t]$ with $i \neq i'$. Let U be the matrix whose column (i,j) is the vector $\mathbf{v}_{i,j}$ (in the lexicographic order $(1,1), (1,2), \dots, (r,t)$). Then $U^\top \cdot U$ is of the form

$$U^\top \cdot U = \begin{pmatrix} \star & I_t & 0 & \dots & 0 \\ I_t & \star & I_t & \ddots & \vdots \\ 0 & I_t & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \star & I_t \\ 0 & \dots & 0 & I_t & \star \end{pmatrix}$$

As $U^\top \cdot U$ contains an $(r-1)t \times (r-1)t$ upper triangular matrix whose diagonal is constant to 1, we obtain that $U^\top \cdot U$ has rank at least $(r-1)t$. It follows that $k \geqslant \mathrm{rk}(U) \geqslant \mathrm{rk}(U^\top \cdot U) \geqslant (r-1)t$. Using Observation 2.4, this proves that $\mathrm{diam}(\mathcal{I}(G)) \geqslant (r-1)t$. □

3. Bounds in terms of degeneracy and strong degeneracy

In this section, we study the relationship between the inversion diameter of a graph G and its degeneracy. Note that the degeneracy of a graph is strongly related to its maximum average degree, as $\delta^*(G) \leqslant \mathrm{Mad}(G) \leqslant 2\delta^*(G)$ holds for every graph G. We start by proving that the inversion diameter of a graph G cannot be upper bounded by a function of $\delta^*(G)$ or $\mathrm{Mad}(G)$.

Proposition 3.1. — *For every positive integer ℓ, there exists a graph G with $\delta^*(G) = 2$ and $\mathrm{diam}(\mathcal{I}(G)) \geqslant \ell$.*

Proof. — Let ℓ be a positive integer. Without loss of generality, $\ell \geqslant 2$. Let G be the graph obtained from $K_{2^{\ell-1}+1}$ by subdividing every edge e once, thereby creating a vertex x_e. We call the vertices of $K_{2^{\ell-1}+1}$ the original vertices of G. One can easily check that $\delta^*(G) = 2$.

Let $\vec{G}_1$ be an orientation of G where $d^-_{\vec{G}_1}(x_e) = 1$ for all $e \in E(K_{2^{\ell-1}+1})$ and let $\vec{G}_2$ be the orientation of G where $d^-_{\vec{G}_2}(x_e) = 2$ for all $e \in E(K_{2^{\ell-1}+1})$. Suppose that there exist $X_1, \dots X_t \subseteq V(G)$ such that $\mathrm{Inv}(\vec{G}_1; X_1, \dots, X_t) = \vec{G}_2$ for some $t < \ell$. By the pigeon-hole principle, there exist two original vertices u and v such that $\{i \mid u \in X_i\} = \{i \mid v \in X_i\}$. It follows that $d^-_{\mathrm{Inv}(\vec{G}_1; X_1, \dots, X_t)}(x_{uv}) = 1$. This is a contradiction to the assumption that $\mathrm{Inv}(\vec{G}_1; X_1, \dots, X_t) = \vec{G}_2$. Hence $\mathrm{diam}(\mathcal{I}(G)) \geqslant \ell$. □

Conversely, the following result shows that the inversion diameter grows with the maximum average degree and the degeneracy.

Theorem 3.2. — $\mathrm{diam}(\mathcal{I}(G)) \geqslant \mathrm{Mad}(G)/2$ *for every graph G.*

Proof. — Set $M = \mathrm{Mad}(G)$. Let H be a subgraph of G such that $\mathrm{Ad}(H) = M$. Let n be the number of vertices of H. The number of edges of H is $\frac{1}{2}Mn$. Then, by Observation 2.2, we deduce $\mathrm{diam}(\mathcal{I}(H)) \geqslant M/2$. Finally, since H is a subgraph of G, we have $\mathrm{diam}(\mathcal{I}(G)) \geqslant \mathrm{diam}(\mathcal{I}(H)) \geqslant M/2$. □

However, a stronger version of degeneracy permits to bound $\mathrm{diam}(\mathcal{I}(G))$. Let us define it.

Let G be a graph and let $<$ be a total ordering on $V(G)$. For every pair u, u' of vertices in G, let $N_{<u'}(u) = \{v \in N(u) \mid v < u'\}$ and $N_{>u'}(u) = \{v \in N(u) \mid v > u'\}$. We simply write $N_<(u)$ for $N_{<u}(u)$ and $N_>(u)$ for $N_{>u}(u)$. The ordering $<$ is *t-strong* if for every $u \in V(G)$

- $|N_<(u)| + \log\left(|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}|\right) < t$ if $N_>(u) \neq \emptyset$, and
- $|N_<(u)| \leqslant t$ otherwise.

A graph is *strongly t-degenerate* if there is a t-strong ordering of its vertices.

We now show that this stronger version of degeneracy is sufficient to provide a bound for the inversion diameter.

THEOREM 3.3. — *Let G be a graph and let t be a positive integer. If G is strongly t-degenerate, then* $\operatorname{diam}(\mathcal{I}(G)) \leqslant t$.

Proof. — Consider a function $\pi\colon E(G) \to \mathbb{F}_2$. By Observation 2.4, it is enough to show the existence of a family $(\mathbf{u})_{u \in V(G)}$ of vectors in $\mathbb{F}_2^t$ such that $\mathbf{u} \cdot \mathbf{v} = \pi(uv)$ for every edge uv of G. Let $<$ be a t-strong ordering of $V(G)$ and let $(v_1, \dots, v_n)$ be the vertices of $V(G)$ as they appear in this order. We iteratively construct vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{F}_2^t$ maintaining the following invariant: for $i \in [n]$

(1) for every $k \in \{i+1, \dots, n\}$, the vectors $(\mathbf{v}_j)_{v_j \in N_{<v_i}(v_k)}$ are linearly independent, and
(2) $\mathbf{v}_j \cdot \mathbf{v}_k = \pi(v_j v_k)$ for every edge $v_j v_k$ of G with $j, k \leqslant i$.

For $i = n$, this will imply that $\operatorname{diam}(\mathcal{I}(G)) \leqslant t$ by Observation 2.4.

For $i = 1$, we take $\mathbf{v}_1 \in \mathbb{F}_2^t \setminus \{\mathbf{0}\}$ arbitrarily. Now suppose $i > 1$ and that $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$ have already been constructed. Since the vectors $(\mathbf{v}_j)_{v_j \in N_<(v_i)}$ are linearly independent, the set of vectors $\mathbf{x} \in \mathbb{F}_2^t$ satisfying $\mathbf{v}_j \cdot \mathbf{x} = \pi(v_i v_j)$ for every $v_j \in N_<(v_i)$ forms an affine space A of dimension at least $t - |N_<(v_i)|$. To satisfy the second condition, we must choose $\mathbf{v}_i$ in A. Note that $|A| \geqslant 2^{t - |N_<(v_i)|} \geqslant 1$ because $<$ is t-strong. Observe that if $N_>(v_i) = \emptyset$, then the first condition trivially holds, and so we can choose any vector in A for $\mathbf{v}_i$.

Henceforth, we may assume that $N_>(v_i) \neq \emptyset$. For every $v_j \in N_>(v_i)$, the first condition of the invariant says that $\mathbf{v}_i$ must be outside the vector space U_j spanned by $(\mathbf{v_k})_{v_k \in N(v_j), k < i}$. Thus it is enough to take $\mathbf{v}_i$ not in $U = \bigcup_{v_j \in N_>(v_i)} U_j$. Now observe that for every $\mathbf{y} \in U$, there is a vertex $v_j \in N_>(v_i)$ and a set $X \subseteq \{v_k \in N(v_j) \mid k < i\}$ such that $\mathbf{y} = \sum_{x \in X} \mathbf{x}$. Hence $|U| \leqslant |\{X \subseteq V(G) \mid \exists\ v_j \in N_>(v_i), X \subseteq N_{<v_i}(v_j)\}|$. Since $<$ is t-strong, we obtain that $|A| \geqslant 2^{t - |N_<(v_i)|} > |\{X \subseteq V(G) \mid \exists\ v_j \in$

$N_>(v_i), X \subseteq N_<(v_j)\}| \geqslant |U|$. We deduce that $A \setminus U \neq \emptyset$ and there is a vector $\mathbf{v}_i$ satisfying the invariant. The statement now follows for $i = n$. □

As a first consequence of this proposition, we prove that graphs of bounded degeneracy have inversion diameter at most logarithmic in the maximum degree.

COROLLARY 3.4. — $\operatorname{diam}(\mathcal{I}(G)) \leqslant 2\delta^*(G) - 1 + \lceil \log(\Delta(G)) \rceil$ *for every graph G with at least one edge.*

Proof. — Let $d = \delta^*(G)$, let $t = 2\delta^*(G) - 1 + \lceil \log(\Delta(G)) \rceil$, and let $<$ be an ordering of $V(G)$ witnessing the fact that $\delta^*(G) \leqslant d$. If $\Delta(G) = 1$, then one can easily check that $\operatorname{diam}(\mathcal{I}(G)) = 1 \leqslant 2d - 1$. Now assume $\Delta(G) \geqslant 2$. For every vertex $u \in V(G)$, we have $|N_<(u)| \leqslant d$ by definition. Moreover, if $N_>(u) \neq \emptyset$ then $|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}| \leqslant 1 + \sum_{v \in N_>(u)} (2^{|N_{<u}(v)|} - 1) \leqslant 1 + \Delta(G)(2^{d-1} - 1) < \Delta(G)2^{d-1}$ (since $\Delta(G) \geqslant 2$). Finally, since $t \geqslant 2d - 1 + \log(\Delta(G))$, the ordering $<$ is t-strong, and the result follows from Theorem 3.3. □

4. Tying the inversion diameter

The aim of this section is to prove the following result.

THEOREM 4.1. — *The inversion diameter is tied to the star chromatic number, the acyclic chromatic number, and the oriented chromatic number.*

Observe that Theorem 4.1 directly implies Theorem 1.2. In order to prove Theorem 4.1, we first give some simple preliminary results on the inversion diameter of forests in Section 4.1. Then, in Section 4.2, we give the proof of Theorem 4.1.

4.1. Inversion diameter of forests

THEOREM 4.2. — *If F is a forest, then* $\operatorname{diam}(\mathcal{I}(F)) \leqslant 2$.

Proof. — Let $\vec{F}_1$ and $\vec{F}_2$ be two orientations of a forest F. Recall that $E_{\neq}$ is the set of edges on which $\vec{F}_1$ and $\vec{F}_2$ disagree. Consider the graph $H_{\neq}$ whose vertices are connected components of $(V(F), E_{\neq})$, and where two such connected components C and C' are connected if and only if there is an edge uv in $E(F)$ with $u \in C$ and $v \in C'$. Observe that $H_{\neq}$ is a minor of F, and so is a forest. In particular, $H_{\neq}$ is bipartite. Let (X'_1, X'_2) be a

bipartition of $H_{\neq}$ and let (X_1, X_2) the partition of $V(F)$ where a vertex is contained in X_i if it is contained in a component of $H_{\neq}$ that belongs to X_i' for $i = 1, 2$. For every edge uv in $E(F)$, we have $uv \in E_{\neq}$ if and only if $\{u, v\} \subseteq X_1$ or $\{u, v\} \subseteq X_2$. Thus $\mathrm{Inv}(\vec{F}_1; (X_1, X_2)) = \vec{F}_2$. □

We next give a simple characterisation of the graphs whose inversion diameter is at most 1. A *star forest* is a graph whose connected components are all isomorphic to a star (including K_1 and K_2).

PROPOSITION 4.3. — *Let G be a graph.*

(i) $\mathrm{diam}(\mathcal{I}(G)) = 0$ *if and only if G has no edges.*

(ii) $\mathrm{diam}(\mathcal{I}(G)) = 1$ *if and only if G is a star forest with at least one edge.*

Proof. — (i) holds trivially. Let us prove (ii). It is easy to see that if G is a star-forest, then $\mathrm{diam}(\mathcal{I}(G)) \leqslant 1$. Assume now that G is not a star forest. Then F has a subgraph which is isomorphic to either the complete graph on three vertices K_3, or the path on four vertices P_4. Now $\mathrm{diam}(\mathcal{I}(K_3)) = 2$ by Theorem 2.5, and $\mathrm{diam}(\mathcal{I}(P_4)) = 2$ since two orientations of P_4 agreeing only on its middle edge are at distance 2. Thus $\mathrm{diam}(\mathcal{I}(G)) \geqslant \min\{\mathrm{diam}(\mathcal{I}(K_3)), \mathrm{diam}(\mathcal{I}(P_4))\} = 2$. □

We now use Theorem 4.2 and Proposition 4.3 to completely determine the inversion diameter of forests.

COROLLARY 4.4. — *If F is a forest, then*

$$\mathrm{diam}(\mathcal{I}(F)) = \begin{cases} 0 & \text{if } F \text{ has no edge,} \\ 1 & \text{if } F \text{ is a star forest with at least one edge,} \\ 2 & \text{otherwise.} \end{cases}$$

4.2. Proof of Theorem 4.1

In this section, we prove that the inversion diameter of a graph is tied to its star chromatic number, its acyclic chromatic number, and its oriented chromatic number. We first show in Corollary 4.6 that the inversion diameter of a graph is bounded by functions of its star chromatic number and its acyclic chromatic number. We then show in Theorem 4.7 that the oriented chromatic number of a graph is bounded by a function of its inversion diameter. Together with some previous results showing that the star chromatic number, the acyclic chromatic number, and the oriented chromatic number are tied, we obtain Theorem 4.1. The main tool we will use here to bound the inversion diameter is the following lemma.

Lemma 4.5. — *Let G be a graph and α and k positive integers. If there exists a proper colouring $c\colon V(G) \to [k]$ such that the subgraph induced by the union of any two colour classes has inversion diameter at most α, then*

$$\operatorname{diam}(\mathcal{I}(G)) \leqslant \alpha\binom{k}{2}.$$

Proof. — Let $\vec{G}_1$ and $\vec{G}_2$ be two orientations of G. Let $c\colon V(G) \to [k]$ be as in the statement. For every $i \in [k]$, let $S_i = c^{-1}(i)$. For every i, j in $[k]$ such that $i < j$, let $V_{i,j} = S_i \cup S_j$. By assumption, there exists a family $\mathcal{X}_{i,j}$ of α subsets of $V_{i,j}$ whose inversion transforms $\vec{G}_1\langle V_{i,j}\rangle$ into $\vec{G}_2\langle V_{i,j}\rangle$. Let $\mathcal{X} = \bigcup_{1\leqslant i<j\leqslant k} \mathcal{X}_{i,j}$. We claim that $\operatorname{Inv}(\vec{G}_1, \mathcal{X}) = \vec{G}_2$.

To see that, consider an arc uv of $\vec{G}_1$, and let $i = c(u)$ and $j = c(v)$. Note that $i \neq j$ because c is a proper colouring. Then uv is reversed only when inverting sets of $\mathcal{X}_{i,j}$. As inverting $\mathcal{X}_{i,j}$ transforms $\vec{G}_1\langle V_{i,j}\rangle$ into $\vec{G}_2\langle V_{i,j}\rangle$, it follows that uv has been inverted if and only if it belongs to $E_{\neq}$. In other words, we obtain $\operatorname{Inv}(\vec{G}_1; \mathcal{X}) = \vec{G}_2$. □

Using Lemma 4.5 and the fact that star-forests and forests have inversion diameter at most 1 and 2 respectively by Corollary 4.4, we deduce the following.

Corollary 4.6. — *Let G be a graph. Then* $\operatorname{diam}(\mathcal{I}(G)) \leqslant \binom{\chi_s(G)}{2}$ *and* $\operatorname{diam}(\mathcal{I}(G)) \leqslant 2\binom{\chi_a(G)}{2}$.

We now show that the oriented chromatic number of a graph is upper bounded by a function of its inversion diameter.

Theorem 4.7. — *Let G be a graph with inversion diameter ℓ. Then $\chi_o(G) \leqslant (2\ell+1)2^\ell$.*

Proof. — Let G be a graph with inversion diameter ℓ. Let $\vec{G}_1$ be an orientation of G. We will show that $\vec{G}_1$ has a homomorphism to an orientation of $K_{(2\ell+1)2^\ell}$.

Since $\chi(G) \leqslant \operatorname{Mad}(G) + 1$ and $\operatorname{Mad}(G) \leqslant 2\ell$ by Theorem 3.2, G has a proper $(2\ell+1)$-colouring $\phi\colon V(G) \to [2\ell+1]$. Let $\vec{G}_0$ be the orientation of G such that $uv \in A(\vec{G}_0)$ if and only if $\phi(u) < \phi(v)$. By definition of ℓ, there exists a sequence $X_1, \dots, X_\ell$ of subsets of $V(G)$ such that $\operatorname{Inv}(\vec{G}_0, \{X_1, \dots, X_\ell\}) = \vec{G}_1$. We define the colouring ψ as follows: for every vertex v in G, $\psi(v) = (\phi(v), \{i \in [\ell], v \in X_i\})$. Clearly, this is a proper $(2\ell+1)2^\ell$-colouring of G. Further, for $c \in [2\ell+1]$ and $I \subseteq [\ell]$, let $V_{(c,I)}$ be the set of vertices coloured (c, I). We claim that, in $\vec{G}_1$, all the arcs between $V_{(c,I)}$ and $V_{(c',I')}$ are either all oriented towards $V_{(c,I)}$ or all oriented towards $V_{(c',I')}$ for all $c, c' \in [2\ell+1]$ and $I, I' \subseteq [\ell]$. To see that, observe that

those arcs are in the same direction in $\vec{G}_0$, that is from $\phi^{-1}(c)$ to $\phi^{-1}(c')$ if $c < c'$, and that they are always reversed together.

Therefore ψ is an oriented colouring of $\vec{G}_1$. As $\vec{G}_1$ was arbitrary, we get $\chi_o(G) \leqslant (2\ell+1)2^\ell$. □

Corollary 4.6, Theorem 4.7, and the fact that χ_s, χ_a, and χ_o are tied [9, 12, 15] imply Theorem 4.1.[(2)]

5. Better bounds in terms of the maximum degree

The fact that the inversion diameter of a graph is upper bounded by a function of its maximum degree follows from Corollary 3.4 as well as the results established in Section 4. By Theorem 2.5, for every positive integer Δ, the complete graph $K_{\Delta+1}$ has maximum degree Δ and $\operatorname{diam}(\mathcal{I}(K_{\Delta+1})) = \Delta$, so the maximum inversion diameter of a graph of maximum degree Δ is at least Δ. We conjecture that this is tight.

Conjecture 5.1. — *For every graph G, we have* $\operatorname{diam}(\mathcal{I}(G)) \leqslant \Delta(G)$.

In this section, we give some evidences for this conjecture to be true. First, in Subsection 5.1, we show that $\operatorname{diam}(\mathcal{I}(G)) \leqslant 2\Delta(G) - 1$ for every graph G with at least one edge, and that $\operatorname{diam}(\mathcal{I}(G)) \leqslant \Delta(G) + o(\Delta(G))$ for every bipartite graph G. In Sections 5.2 and 5.3, we prove Conjecture 5.1 when $\Delta = 2$, and that graphs of maximum degree 3 have inversion diameter at most 4.

5.1. Bounds for arbitrary maximum degree

In this subsection, we prove the following weakening of Conjecture 5.1.

Theorem 5.2. — *For every graph G with at least one edge,*

$$\operatorname{diam}(\mathcal{I}(G)) \leqslant 2\Delta(G) - 1.$$

We need the following lemma.

Lemma 5.3. — *For every positive Δ with $\Delta \geqslant 2$, for every integer d with $0 \leqslant d \leqslant \Delta$,*

$$2^{2\Delta-1-d} > 1 + (\Delta - d)(2^{\Delta-1} - 1).$$

(2) Slightly better bounds on the inversion diameter in terms of χ_s, χ_a, χ_o can be found in a longer version of this paper, available at https://arxiv.org/pdf/2405.04119v2.

Proof. — If $\Delta = d$, we have $2^{2\Delta-1-d} = 2^{\Delta-1} \geqslant 2 > 1 = 1 + (\Delta-d)(2^{\Delta-1}-1)$. Otherwise, as $k < 2^k$ holds for every nonnegative integer k, we have

$$\begin{aligned} 1 + (\Delta - d)(2^{\Delta-1} - 1) &\leqslant (\Delta - d)2^{\Delta-1} \\ &< 2^{\Delta-d}2^{\Delta-1} \\ &= 2^{2\Delta-1-d}. \end{aligned}$$

□

Proof of Theorem 5.2. — If $\Delta(G) = 1$, then $\operatorname{diam}(\mathcal{I}(G)) = 1$. Otherwise, let $\Delta = \Delta(G)$ and let $<$ be an arbitrary ordering of $V(G)$. Now consider some $u \in V(G)$ and let $d = |N_<(u)|$. If $N_>(u) \neq \emptyset$, then we have

$$\begin{aligned} |N_<(u)| + \log\left(|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_<(v)\}|\right) \\ \leqslant d + \log(1 + (\Delta - d)(2^{\Delta-1} - 1)) \\ < 2\Delta - 1 \text{ by Lemma 5.3.} \end{aligned}$$

If $N_>(u) = \emptyset$, then $|N_<(u)| \leqslant \Delta \leqslant 2\Delta - 1$. Hence G is strongly $(2\Delta-1)$-degenerate, so $\operatorname{diam}(\mathcal{I}(G)) \leqslant 2\Delta - 1$ by Theorem 3.3. □

For bipartite graphs, Theorem 5.2 can be improved as follows.

THEOREM 5.4. — $\operatorname{diam}(\mathcal{I}(G)) \leqslant \Delta + \lceil \log \Delta \rceil - 1$ *for every bipartite graph G of maximum degree $\Delta \geqslant 2$.*

Proof. — Let $t = \Delta + \lceil \log \Delta \rceil - 1$. Let G be a bipartite graph of maximum degree Δ and let (A, B) be a bipartition of G. Let $<$ be an ordering of $V(G)$ such that $a < b$ for every $a \in A$ and $b \in B$. For every $a \in A$ we have $N_<(a) = \emptyset$ and $|\{X \subseteq V(G) \mid \exists\, v \in N_>(a), X \subseteq N_{<a}(v)\}| \leqslant 1 + \Delta(2^{\Delta-1} - 1) < \Delta 2^{\Delta-1} \leqslant 2^t$. For every $b \in B$, we have $|N_<(b)| \leqslant \Delta \leqslant t$ and $|\{X \subseteq V(G) \mid \exists\, v \in N_>(b), X \subseteq N_<(v)\}| = 0$. Hence $<$ is a t-strong ordering of G. Thus, by Theorem 3.3, we have $\operatorname{diam}(\mathcal{I}(G)) \leqslant t$. □

5.2. Maximum degree at most 2

We here prove the following result for graphs of maximum degree at most 2.

THEOREM 5.5. — *Let G be a graph. If $\Delta(G) \leqslant 2$, then* $\operatorname{diam}(\mathcal{I}(G)) \leqslant 2$.

Proof. — By possibly adding some edges, we can assume that every connected component of G is a cycle or a single edge. Moreover, by Observation 2.1 (2), and since $\operatorname{diam}(\mathcal{I}(K_2)) = 1$, it is enough to prove the statement for cycles.

Let $\vec{C}_1$ and $\vec{C}_2$ be two orientations of a cycle C. First assume that $E_=$ contains two consecutive edges. Let x be the common vertex of those two edges. By Theorem 4.2, there are two sets X_1 and X_2 of $V(C-x)$ such that $\mathrm{Inv}(\vec{C}_1 - x; X_1, X_2) = \vec{C}_2 - x$. Then $\mathrm{Inv}(\vec{C}_1; X_1, X_2) = \vec{C}_2$.

Now assume that there are no two consecutive edges in $E_=$. If $E_= = \emptyset$, then inverting $V(C)$ transforms $\vec{C}_1$ into $\vec{C}_2$. Henceforth, we may assume that $E_= \neq \emptyset$. Let $P_1, P_2, \dots, P_t$ be the connected components of $C \setminus E_=$. If t is even, let $X_1 = \bigcup_{i=1}^{t/2} V(P_{2i-1})$ and $X_2 = \bigcup_{i=1}^{t/2} V(P_{2i})$; we have $\mathrm{Inv}(\vec{C}_1; X_1, X_2) = \vec{C}_2$. If t is odd, set $s = \frac{t-1}{2}$ and let $X_1 = \bigcup_{i=0}^{s} V(P_{2i+1})$ and $X_2 = \{x, y\} \cup \bigcup_{i=1}^{s} V(P_{2i})$ where xy is the edge between P_1 and P_t; we have $\mathrm{Inv}(\vec{C}_1; X_1, X_2) = \vec{C}_2$. □

5.3. Subcubic graphs

For subcubic graphs, we can slightly improve on Theorem 5.2. First we need some preliminaries.

Let G be a subcubic multigraph, $\pi\colon E(G) \to \mathbb{F}_2$ a mapping, and let $\sigma = (v_1, \dots, v_n)$ be an ordering of $V(G)$. We say that a vertex v_i is *bad* in (σ, π) if there exist j_1, j_2, j_3 such that $j_1 \leqslant j_2 < i < j_3$, $v_{j_1}v_i, v_{j_2}v_i$, and $v_{j_3}v_i$ are distinct edges of $E(G)$, and $\pi(v_{j_1}v_i) = \pi(v_{j_2}v_i) = 0$ and $\pi(v_iv_{j_3}) = 1$. Observe that this includes the case that $j_1 = j_2$ where the two corresponding edges are parallel. When π is clear from the context, we will simply say that v is *bad* in σ. The ordering σ is *good* if it has no bad vertex. The following result is the main ingredient in the proof of Theorem 5.8.

Lemma 5.6. — *Let G be a subcubic multigraph and $\pi\colon E(G) \to \mathbb{F}_2$ a mapping. Then there is a good ordering of $V(G)$.*

Proof. — Suppose otherwise and let (G, π) be a counterexample to the statement with $|V(G)|$ minimum. We say that an edge $e = uv \in E(G)$ is *critical* if $\pi(e) = 1, d_G(u) = 3$, $d_G(v) = 3$, $E(G) \setminus e$ does not contain an edge which is parallel to e, and $\pi(e') = 0$ for every $e' \in E(G) \setminus e$ which is incident to at least one of u and v.

Claim 5.7. — *No edge in $E(G)$ is critical.*

Proof of the claim. Suppose otherwise, so $E(G)$ contains a critical edge $e_0 = uv$. Let ut_1 and ut_2, and vw_1 and vw_2 be the edges distinct from e incident to u and v, respectively. Observe that $\{u, v\} \cap \{t_1, t_2, w_1, w_2\} = \emptyset$ by assumption. In the following, we distinguish several cases on the structure

of these neighbourhoods, and obtain contradictions which eventually prove the claim. By symmetry, we may suppose that one of the following four cases occurs.

Case 1. — *$t_1 = t_2$ and $w_1 = w_2$.*

Let G' be obtained from $G - \{u, v\}$ by adding two edges e_1, e_2 linking t_1 and w_1. We further define $\pi' \colon E(G') \to \mathbb{F}_2$ by $\pi'(e_1) = \pi'(e_2) = 0$ and $\pi'(e) = \pi(e)$ for all $e \in E(G) \cap E(G')$. As G' is subcubic and smaller than G, there exists a good ordering $\sigma' = (v_1, \dots, v_{n-2})$ of $V(G')$ with respect to π'. Let $i, j \in [n-2]$ such that $v_i = t_1$ and $v_j = w_1$. Since G is subcubic, we have $i \neq j$ and by symmetry, we may suppose that $i < j$. Now consider the ordering $\sigma = (v_1, \dots, v_i, v, u, v_{i+1}, \dots, v_{n-2})$. We obtain that none of t_1, u, and v are bad in σ. Furthermore, w_1 is not bad in σ as σ' is a good ordering of $V(G')$ with respect to π'. Finally, no vertex in $V(G) \setminus \{u, v, t_1, w_1\}$ is bad in σ as σ' is a good ordering of $V(G')$ with respect to π'. It follows that σ is a good ordering of $V(G)$ with respect to π, a contradiction to the choice of (G, π).

Case 2. — *$t_2 = w_1 = w_2$.*

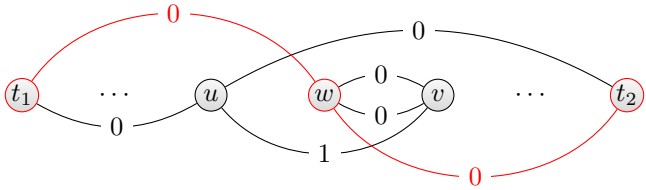
Let $G' = G - \{u, v, t_2\}$ and $\pi' = \pi|_{E(G')}$. As G' is subcubic and smaller than G, there exists a good ordering $\sigma' = (v_1, \dots, v_{n-3})$ of $V(G')$ with respect to π'. Now consider the ordering $\sigma = (t_1, u, v, t_2, v_1, \dots, v_{n-3})$. We obtain that none of t_1, t_2, u, and v are bad in σ. Further, no vertex in $V(G) \setminus \{u, v, t_1, t_2\}$ is bad in σ as σ' is a good ordering of $V(G')$ with respect to π'. It follows that σ is a good ordering of $V(G)$ with respect to π, a contradiction to the choice of (G, π).

Case 3. — *$w_1 = w_2$ and t_1, t_2, and w_1 are pairwise distinct.*

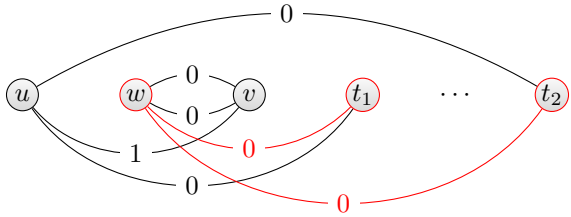
Let G' be obtained from $G - \{u, v\}$ by adding two edges: e_1 with endvertices t_1, w_1, and e_2 with endvertices t_2, w_1. We further define $\pi' \colon E(G') \to \mathbb{F}_2$ by $\pi'(e_1) = \pi'(e_2) = 0$ and $\pi'(e) = \pi(e)$ for all $e \in E(G) \cap E(G')$. As G' is subcubic and smaller than G, there exists a good ordering $\sigma' = (v_1, \dots, v_{n-2})$ of $V(G')$ with respect to π'. Let $i, j, k \in [n-2]$ be such that $v_i = t_1$, $v_j = t_2$ and $v_k = w_1$. Note that i, j, and k are pairwise distinct. By symmetry, we may suppose that $i < j$.

(a) If $i < k < j$, then let $\sigma = (v_1, \dots, v_{k-1}, u, w_1, v, v_{k+1}, \dots, v_{n-2})$.
(b) If $k < i < j$, then let $\sigma = (v_1, \dots, v_{k-1}, u, w_1, v, v_{k+1}, \dots, v_{n-2})$.
(c) If $i < j < k$, then let $\sigma = (v_1, \dots, v_{k-1}, v, u, w_1, v_{k+1}, \dots, v_{n-2})$.

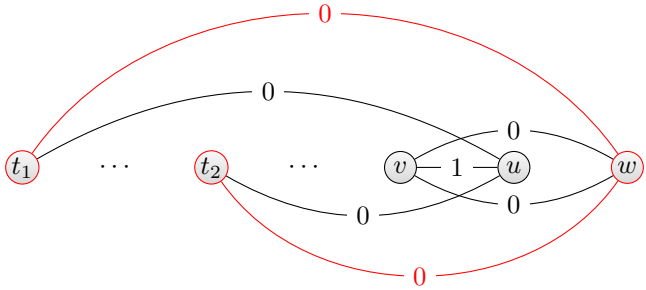
All of these cases are illustrated in Figure 5.1. Similar arguments as above show that σ is a good ordering of G with respect to π. This contradicts the choice of (G, π).



(a) Case 3 (a): $i < k < j$.



(b) Case 3 (b): $k < i < j$.



(c) Case 3 (c): $i < j < k$.

Figure 5.1. Some cases in the proof of Lemma 5.6. The vertices are placed from left to right according to σ. The edges in $E(G') \setminus E(G)$ and the vertices in $V(G')$ are depicted in red.

Case 4. — *$t_1 \neq t_2$ and $w_1 \neq w_2$.*

Let G' be obtained from $G - \{u, v\}$ by adding two edges: e_1 with endvertices t_1, t_2, and e_2 with endvertices w_1, w_2. We further define $\pi' \colon E(G') \to$

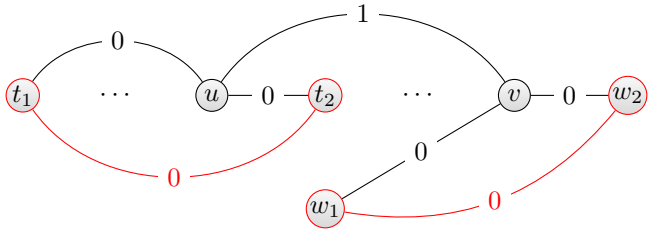


Figure 5.2. The case $i < j, k < l, j < l$ in the proof of Lemma 5.6, with vertices placed from left to right according to σ. Since the relative position of w_1 with t_1 and t_2 is unknown, w_1 is placed below. The edges in $E(G') \setminus E(G)$ and the vertices in $V(G')$ are depicted in red.

$\mathbb{F}_2$ by $\pi'(e_1) = \pi'(e_2) = 0$ and $\pi'(e) = \pi(e)$ for all $e \in E(G) \cap E(G')$. As G' is subcubic and smaller than G, there exists a good ordering $\sigma' = (v_1, \dots, v_{n-2})$ of $V(G')$ with respect to π'. Let $i, j, k, \ell \in [n-2]$ be such that $v_i = t_1$, $v_j = t_2, v_k = w_1$, and $v_\ell = w_2$. Without loss of generality, we may suppose that $i < j, k < \ell$, and $j \leqslant \ell$. If $j < \ell$, then let

$$\sigma = (v_1, \dots, v_{j-1}, u, t_2, v_{j+1}, \dots, v_{\ell-1}, v, w_2, v_{\ell+1}, \dots, v_{n-2}).$$

This case is illustrated in Figure 5.2. If $j = \ell$, then observe that $t_2 = w_2$. Let

$$\sigma = (v_1, \dots, v_{j-1}, u, v, w_2, v_{j+1}, \dots, v_{n-2}).$$

In either case, we obtain that none of t_1, u, w_1, and v is bad in σ. Furthermore, t_2 and w_2 are not bad in σ as σ' is a good ordering of $V(G')$ with respect to π'. Finally, no vertex in $V(G) \setminus \{u, v, t_1, w_1\}$ is bad in σ as σ' is a good ordering of $V(G')$ with respect to π'. It follows that σ is a good ordering of $V(G)$ with respect to π, a contradiction to the choice of (G, π).

This finishes the proof of the claim. ◊

Let V_0 be the set of vertices $v \in V(G)$ with $d_G(v) = 3$ that are incident to exactly one edge in $\pi^{-1}(1)$. Now consider an ordering σ of $V(G)$ in which all vertices in V_0 are preceded by all vertices in $V(G) \setminus V_0$. Clearly, no vertex in $V(G) \setminus V_0$ is bad in σ. Now consider some vertex $v \in V_0$ and let w be its unique neighbour such that $\pi(vw) = 1$. By Claim 5.7, we obtain that $w \in V(G) \setminus V_0$. By construction, it follows that $w < v$ with respect to σ. Hence v is not bad with respect to σ, and it follows that σ is a good ordering of G with respect to π. This contradicts the choice of (G, π). □

We now give the main result of this section. The approach is very similar to the one of Theorem 3.3. We use the ordering whose existence is guaranteed by Lemma 5.6 and make some finer case distinctions.

THEOREM 5.8. — *Let G be a graph. If $\Delta(G) \leqslant 3$, then $\operatorname{diam}(\mathcal{I}(G)) \leqslant 4$.*

Proof. — Let G be a graph of maximum degree at most 3. Consider a function $\pi\colon E(G) \to \mathbb{F}_2$. We show that there is a family $(\mathbf{u})_{u\in V(G)}$ of vectors in $\mathbb{F}_2^4$ such that $\pi(uv) = \mathbf{u}\cdot\mathbf{v}$ for every edge $uv \in E(G)$. If there is a vertex u incident only to edges e with $\pi(e) = 0$, then we apply induction on $G - u$ and set $\mathbf{u} = \mathbf{0}$. Now we assume that every vertex of G is incident to at least one edge e with $\pi(e) = 1$.

By Lemma 5.6, there is a good ordering $\sigma = (v_1, \dots, v_n)$ of G with respect to π. We iteratively construct $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{F}_2^4$ with the following property for every $i \in [n]$:

(1) for every $k > i$, $(\mathbf{v}_j)_{j\leqslant i, v_j\in N(v_k)}$ are linearly independent, and
(2) for every edge $v_jv_{j'}$ with $j, j' \leqslant i$, $\mathbf{v}_j \cdot \mathbf{v}_{j'} = \pi(v_jv_{j'})$.

Initially, it suffices to take $\mathbf{v}_1 \neq \mathbf{0}$. Now suppose that $\mathbf{v}_1, \dots, \mathbf{v}_i$ satisfy this invariant for some $i \in [n-1]$. We denote by d the number of neighbours of v_{i+1} in $\{v_1, \dots, v_i\}$. Condition (1) forbids for each neighbour v_k of v_{i+1} with $k > i+1$ a vector space of dimension at most $3 - 1 = 2$, and so in total, their union U_1 has size at most $1 + (3-d)(2^2-1)$ and contains $\mathbf{0}$. Condition (2) allows at least 2^{4-d} values of $\mathbf{v}_{i+1}$. Indeed, as $(\mathbf{v}_j)_{v_j\in N(v_{i+1}), j\leqslant i}$ are linearly independent, the affine space U_2 of the solutions of the system of equations, $\mathbf{v}_j \cdot \mathbf{v}_{i+1} = \pi(v_jv_{i+1})$, for $v_j \in N(v_{i+1})$ with $j \leqslant i$, has dimension at least $4 - d$. It is enough to show that there is a vector in $U_2 \setminus U_1$. If $d = 0$, we have $|U_2| = 2^4 > 1 + 3\times 3 \geqslant |U_1|$. If $d = 1$, we obtain $|U_2| = 2^3 > 1 + 2 \times 3 \geqslant |U_1|$. If $d = 3$, then $|U_2| = 2 > 1 \geqslant |U_1|$. As G is subcubic, it remains to consider the case $d = 2$. If $d_G(v_{i+1}) = 2$, we obtain $|U_1| = 0 < 2^2 = |U_2|$. We may hence suppose that there exists a unique neighbour v_k of v_{i+1} with $k > i+1$. If v_k has a higher-indexed neighbour, then $|U_1| \leqslant 2^1 < 2^2 = |U_2|$. We may hence suppose that v_k has no higher-indexed neighbour. As v_{i+1} is not bad in σ and v_{i+1} is incident to at least one edge in $\pi^{-1}(1)$, v_{i+1} has a neighbour v_j with $j < i+1$ such that $\pi(v_jv_{i+1}) = 1$. This implies that $\mathbf{0} \notin U_2$, and so $|U_2 \setminus U_1| = |U_2 \setminus (U_1 \setminus \{\mathbf{0}\})| \geqslant 2^2 - 3 > 0$.

In each case, we found $\mathbf{v}_{i+1}$ such that $\mathbf{v}_1, \dots, \mathbf{v}_{i+1}$ satisfies the invariant. In the end of the process, we have $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that for every edge v_iv_j, $\mathbf{v}_i \cdot \mathbf{v}_j = \pi(v_iv_j)$. Using Observation 2.4, this proves the theorem. □

It follows from Theorem 2.5 applied to K_4 that the bound in Theorem 5.8 cannot be improved further than 3. Quite surprisingly, the following result shows that K_4 is not a particular example with this property. Indeed, the inversion diameter of any cubic graph is at least 3.

THEOREM 5.9. — *Let G be a graph. If $\delta(G) \geqslant 3$, then $\operatorname{diam}(\mathcal{I}(G)) \geqslant 3$.*

Hence, if Conjecture 5.1 holds for $\Delta(G) = 3$, then the inversion diameter of every cubic graph is exactly 3.

Theorem 5.9 easily follows from the following two lemmas, the latter of which will be reused in Section 7.

LEMMA 5.10. — *Let G be a graph. If $\delta(G) \geqslant 3$, then G contains an even cycle.*

Proof. — Let P be a path of maximal length in G and let y be an endvertex of P, and let z be its predecessor along P. As P is maximal, $N(y) \subseteq V(P)$. For any two vertices u, v of P, we denote by $P[u, v]$ the subpath of P with endvertices u and v. Let $x_1, x_2 \in V(P)$ be such that $N(y) = \{x_1, x_2, z\}$ with x_2 closer to y than x_1 in P. Consider the three cycles $C_1 = P[x_1, x_2] \cup x_2yx_1$, $C_2 = P[x_2, y] \cup yx_2$ and $C_3 = P[x_1, y] \cup yx_1$. Observe that $|V(C_1)| + |V(C_2)| = |V(C_3)| + 2$, thus at least one of $|V(C_1)|$, $|V(C_2)|$ or $|V(C_3)|$ is even, and so G contains an even cycle. □

LEMMA 5.11. — *Let G be a graph that contains an even cycle C with $d_G(v) \geqslant 3$ for all $v \in V(C)$. Then $\operatorname{diam}(\mathcal{I}(G)) \geqslant 3$.*

Proof. — Let $C = v_1v_2 \cdots v_gv_1$. Let $\vec{G}_1, \vec{G}_2$ be two orientations of G that agree on $E(C) \setminus \{v_gv_1\}$, and disagree on all other edges in $E(G)$. Suppose, for the sake of a contradiction, that there is an inversion sequence of (X_1, X_2) that transforms $\vec{G}_1$ into $\vec{G}_2$. As $d_G(v) \geqslant 3$ for all $v \in V(C)$, every vertex in $V(C)$ is incident to an edge of $E_{\neq}$. Hence $X_1 \cup X_2$ covers $V(C)$.

If $v_i \in X_1 \cap X_2$ for some $i \in \{1, \dots, g-1\}$, then $v_{i+1} \in X_1 \cap X_2$ as $v_iv_{i+1} \in E_{=}$. Similarly, if $v_i \in X_1 \cap X_2$ for some $i \in \{2, \dots, g\}$, then $v_{i-1} \in X_1 \cap X_2$. Repeatedly applying these arguments, if there is some $i \in [g]$ with $v_i \in X_1 \cap X_2$, then $v_1, v_2, \dots, v_g \in X_1 \cap X_2$. However, it then follows that v_gv_1 is not reversed, a contradiction. Now suppose that no vertex in $V(C)$ belongs to $X_1 \cap X_2$.

Without loss of generality, suppose that $v_1 \in X_1$. For every $i \in [g-1]$ and $j \in [2]$, we have $v_i \in X_j$ if and only if $v_{i+1} \in X_{3-j}$ as $v_iv_{i+1} \in E_{=}$. Thus, by induction, $v_i \in X_1$ if i odd and $v_i \in X_2$ if i even. In particular, $v_g \in X_2$, but then v_gv_1 is not reversed, a contradiction. □

Lemma 5.10 and Lemma 5.11 directly imply Theorem 5.9.

6. Bounds in terms of treewidth

In this section, we consider the maximum inversion diameter M_t of a graph with treewidth at most t. The graphs with treewidth 1 are the forests, so $M_t = 2$ by Corollary 4.4. We shall prove $t+2 \leqslant M_t \leqslant 2t$ for all $t \geqslant 2$. We start by proving the upper bound $M_t \leqslant 2t$. First, we provide a definition of treewidth.

The treewidth of a graph G, denoted by $\mathrm{tw}(G)$, can be defined as follows. (See for example [6] for further background on treewidth.) For every positive integer t, we define inductively a *t-tree* to be a graph constructed as follows. The complete graph K_{t+1} is a t-tree; if H is a t-tree, then any graph obtained from H by adding a new vertex v and connecting v to all vertices in a t-clique in H is a t-tree. Then the *treewidth* of a graph G with $E(G) \neq \emptyset$ is the minimum t such that G is the subgraph of some t-tree. For completeness, we set the treewidth of a graph with no edge to be 0.

Here are a few facts about treewidth, which we will frequently use.

Observation 6.1.

(i) Let G be a graph, and let t be the treewidth of G. There is no clique of size $t+2$ in G.

(ii) Let G be a graph of treewidth at least 2 and let $u, v, w \in V(G)$ with $uv, uw, vw \in E(G)$ and $d_G(u) = 2$. Then $\mathrm{tw}(G) = \mathrm{tw}(G-u)$.

(iii) Let G_1, G_2 be disjoint graphs, $v_i \in V(G_i)$ for $i \in [2]$, and let G be obtained from G_1 and G_2 by identifying v_1 and v_2. Then $\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}$.

(iv) Let G be a graph and $v \in V(G)$. Then $\mathrm{tw}(G-v) \geqslant \mathrm{tw}(G) - 1$. Moreover, if v is adjacent in G to all the vertices in $V(G)-v$, then equality holds.

We can now prove our upper bound $M_t \leqslant 2t$.

Theorem 6.2. — *$\mathrm{diam}(\mathcal{I}(G)) \leqslant 2\,\mathrm{tw}(G)$ for every graph G.*

Proof. — Let G be a graph of treewidth t. By the first bullet point of Observation 2.1, we can assume without loss of generality that G is a t-tree. Hence, there is a total ordering $<$ of $V(G)$ such that $N_<(v)$ induces a clique of size at most t in G, for every $v \in V(G)$. We claim that $<$ is a $2t$-strong ordering. Let $u \in V(G)$. First suppose that $N_>(u) \neq \emptyset$. Then, for every $v \in N_>(u)$, we have $N(v) \cap \{w \in V(G) \mid w < u\} \subseteq N_<(u)$. Moreover, if

$N(v) \cap \{w \in V(G) \mid w < v\} = N_<(u)$, then $N_<(u) \cup \{u, w\}$ induce a clique of size $t+2$ in G, a contradiction to Observation 6.1 (i). We deduce that

$$|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}| \leqslant 2^t - 1$$

and so

$$|N_<(u)| + \log\left(|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}|\right) \leqslant t + \log(2^t - 1) < 2t.$$

If $N_>(u) = \emptyset$, then $|N_<(u)| \leqslant t \leqslant 2t$. Hence G is strongly $2t$-degenerate and the result follows from Theorem 3.3. □

For the lower bound, observe that $M_t \geqslant t$ can be easily proved by combining Theorem 2.5 applied to the complete graph K_t with the well-known fact that $\mathrm{tw}(K_t) = t-1$ for every $t \geqslant 2$. Somewhat surprisingly, it turns out that complete graphs are not tight examples. Indeed, we slightly improve this lower bound, showing that $M_t \geqslant t+2$ for every $t \geqslant 2$.

We need the following definition. Given a vector $\mathbf{v}$ in $\mathbb{F}_2^t$ for some positive integer t, its *support* is the number of its coordinates whose value is 1.

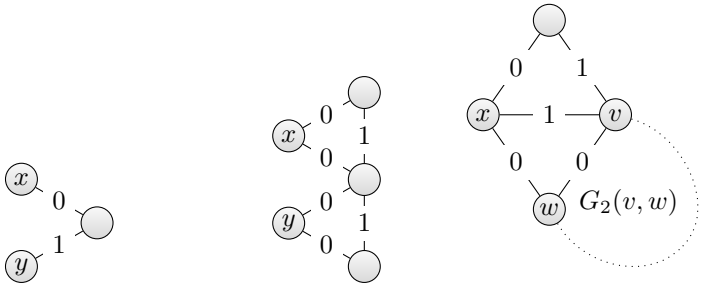
The proof of the lower bound on M_t for $t \geqslant 2$ is by induction on t. The following construction for $t = 2$ will serve as the base case.

Proposition 6.3. — *There is a graph G with treewidth 2 such that* $\mathrm{diam}(\mathcal{I}(G)) \geqslant 4$.

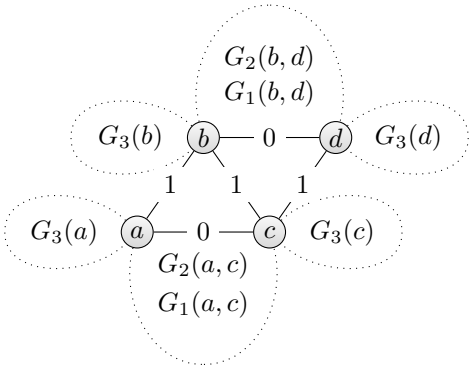
Proof. — Consider the gadgets $G_1(x,y), G_2(x,y)$ and $G_3(x)$ depicted in Figure 6.1. The label functions π_1, π_2, π_3 mapping the edges of $G_1(x,y)$, $G_2(x,y)$, and $G_3(x)$, respectively, to $\mathbb{F}_2$ can also be found in Figure 6.1. For every $i \in [3]$, we say that a family $(\mathbf{u})_{u \in V(G_i)}$ of vectors over $\mathbb{F}_2$ *satisfies* (G_i, π_i) if $\pi_i(uv) = \mathbf{u} \cdot \mathbf{v}$ for every $uv \in E(G_i)$. The following observations can be easily checked.

- For every family $(\mathbf{u})_{u \in V(G_1(x,y))}$ of vectors in $\mathbb{F}_2^3$ satisfying $(G_1(x,y), \pi_1)$, we have $\mathbf{x} \neq \mathbf{y}$.
- For every family $(\mathbf{u})_{u \in V(G_2(x,y))}$ of vectors in $\mathbb{F}_2^3$ satisfying $(G_2(x,y), \pi_2)$, if $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, then $\mathbf{x} + \mathbf{y} \neq (1,1,1)$.
- For every family $(\mathbf{u})_{u \in V(G_3(x))}$ of vectors in $\mathbb{F}_2^3$ satisfying $(G_3(x), \pi_3)$, we have $\mathbf{x} \neq (1,1,1)$.

Now consider the graph G with the function $\pi\colon E(G) \to \mathbb{F}_2$ depicted on Figure 6.1d. We claim that $\mathrm{diam}(\mathcal{I}(G)) > 3$. Suppose for the sake of a contradiction that there is a family $(\mathbf{u})_{u \in V(G_1(x,y))}$ of vectors in $\mathbb{F}_2^3$ satisfying the constraints $\mathbf{u} \cdot \mathbf{v} = \pi(uv)$ for every edge uv. By the previous observations, we have $(1,1,1) \notin \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, $\mathbf{a} \neq \mathbf{c}$ and $\mathbf{b} \neq \mathbf{d}$. Since every



(a) The gadget $G_1(x, y)$. *(b)* The gadget $G_2(x, y)$. *(c)* The gadget $G_3(x)$.



(d) The graph G.

Figure 6.1. Construction of a graph with treewidth 2 and inversion diameter 4.

vertex in $\{a, b, c, d\}$ is incident to an edge e with $\pi(e) = 1$, we also have $(0, 0, 0) \notin \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

By the above observations, we obtain that the support of $\mathbf{b}$ is either 1 or 2. If $\mathbf{b}$ has support 1, say $\mathbf{b} = (1, 0, 0)$, then $\mathbf{c} \in \{(1, 1, 0), (1, 0, 1), (1, 0, 0)\}$ as $\pi(bc) = 1$ and $\mathbf{c} \neq (1, 1, 1)$. If $\mathbf{c} = (1, 0, 0)$, we obtain $0 = \pi(bd) = \mathbf{b} \cdot \mathbf{d} = \mathbf{c} \cdot \mathbf{d} = \pi(cd) = 1$, a contradiction. This yields $\mathbf{c} \in \{(1, 1, 0), (1, 0, 1)\}$ and so $\mathbf{c}$ has support 2. Symmetrically, if $\mathbf{c}$ has support 1, then $\mathbf{b}$ has support 2. We conclude that either $\mathbf{b}$ or $\mathbf{c}$ has support 2. Without loss of generality, suppose that $\mathbf{b}$ has support 2 and $\mathbf{b} = (1, 1, 0)$. Since $\mathbf{b} \cdot \mathbf{d} = 0$, we have $\mathbf{d} \in \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$. However, by the above observations, we have $\mathbf{d} \notin \{(0, 0, 0), (1, 1, 1)\}$, $\mathbf{b} \neq \mathbf{d}$, and $\mathbf{b} + \mathbf{d} \neq (1, 1, 1)$, and so $\mathbf{d} \notin \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$, a contradiction.

Thus $\mathrm{diam}(\mathcal{I}(G)) > 3$ by Observation 2.4. Further, as G is not a tree and repeatedly applying Observation 6.1 (ii) and (iii), we obtain that G has treewidth 2. This proves the proposition. □

We now show that more generally there are graphs of treewidth t and inversion diameter $t+2$, for every $t \geqslant 2$. Given a vector $\mathbf{u} \in \mathbb{F}_2^t$, we denote by $\mathbf{u}^\perp$ the subspace $\{\mathbf{v} \in \mathbb{F}_2^t \mid \mathbf{u} \cdot \mathbf{v} = 0\}$.

In order to extend the lower bound for arbitrary values of t, we need the following result. For some positive integer t, a basis $(\mathbf{x}_1, \dots, \mathbf{x}_t)$ of $\mathbb{F}_2^t$ is called *orthonormal* if $\mathbf{x_i} \cdot \mathbf{x_i} = 1$ holds for all $i \in [t]$ and $\mathbf{x_i} \cdot \mathbf{x_j} = 0$ holds for all distinct $i, j \in [t]$.

Lemma 6.4. — *Let t be an integer such that $t \geqslant 2$. Let $\mathbf{u}$ be a vector in $\mathbb{F}_2^t$ whose support is odd and smaller than t. Then the subspace $\mathbf{u}^\perp$ has an orthonormal basis. In particular, there are at most $2^{t-1}+1$ vectors $\mathbf{u}$ in $\mathbb{F}_2^t$ such that $\mathbf{u}^\perp$ does not admit an orthonormal basis.*

Proof. — Let ℓ be the support of $\mathbf{u}$. Without loss of generality, $\mathbf{u} = (1, \dots, 1, 0, \dots, 0)$. Let J_ℓ be the $(\ell+1) \times (\ell+1)$ matrix all of whose entries are 1 except those on the anti-diagonal, that is $(J_\ell)_{i,j} = 0$ if and only if $i = \ell + 2 - j$ for every $i, j \in [\ell+1]$. Since ℓ is odd, we have $J_\ell^2 = I_{\ell+1}$. Then the matrix

$$M = \begin{pmatrix} J_\ell & 0 \\ 0 & I_{t-\ell-1} \end{pmatrix}$$

is such that $M^2 = I_t$ and its first column is $\mathbf{u}$. Hence the columns of M form an orthonormal basis of $\mathbb{F}_2^t$ containing $\mathbf{u}$. Therefore the family of columns obtained from the columns of M by deleting the first one forms an orthonormal basis of $\mathbf{u}^\perp$. □

We are now ready to prove $M_t \geqslant t+2$.

Theorem 6.5. — *For every $t \geqslant 2$, there is a graph G_t of treewidth t such that $\mathrm{diam}(\mathcal{I}(G)) \geqslant t+2$.*

Proof. — We proceed by induction on t. For $t = 2$, the result follows from Proposition 6.3. Now assume $t \geqslant 3$. By the induction hypothesis and Observation 2.4, there is a graph G_{t-1} with $\mathrm{tw}(G_{t-1}) = t-1$ and a function $\pi_{t-1} \colon E(G_{t-1}) \to \mathbb{F}_2$ such that there is no family of vectors $(\mathbf{z})_{z \in V(G_{t-1})}$ in $\mathbb{F}_2^t$ such that $\pi_{t-1}(yz) = \mathbf{y} \cdot \mathbf{z}$ for every edge $yz \in E(G_{t-1})$.

We now construct G_t as follows. We first let $V(G_t)$ contain t vertices $v_1, \dots, v_t$. Then, for every $\mathbf{x} \in \mathbb{F}_2^t$, we add a vertex $w_\mathbf{x}$ adjacent to all vertices in $\{v_1, \dots, v_t\}$ and we set $\pi(v_i w_\mathbf{x}) = \mathbf{x}_i$ for every $i \in [t]$. Moreover, for every $i \in [t]$ and $\mathbf{x} \in \mathbb{F}_2^t$, we add a vertex $a_{i,\mathbf{x}}$ adjacent to v_i and $w_\mathbf{x}$,

and set $\pi(a_{i,\mathbf{x}}v_i) = 0$ and $\pi(a_{i,\mathbf{x}}w_{\mathbf{x}}) = 1$. Finally, for every $b \in B$, we add a copy G_b of G_{t-1} with the labels π_{t-1} and, for all $z \in V(G_b)$, we add the edge bz and set $\pi(bz) = 0$ where $B = \{v_1, \dots, v_t\} \cup \{w_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{F}_2^t}$.

We first show that the treewidth of G_t is as desired.

Claim 6.6. — $\mathrm{tw}(G_t) = t$.

Proof. Because $\mathrm{tw}(G_{t-1}) = t - 1$ and by Observation 6.1 (iv), we have $\mathrm{tw}(G_t[V(G_b) \cup \{b\}]) = t$ for all $b \in B$. By Observation 6.1 (iii), we obtain $\mathrm{tw}(G_t) \geqslant t$ and $\mathrm{tw}(G_t) = \max\{t, \mathrm{tw}(G_t - \bigcup_{b\in B} V(G_b))\}$. Next, as $G_t[B]$ is not a tree, we obtain that $\mathrm{tw}(G_t - \bigcup_{b\in B} V(G_b)) = \mathrm{tw}(G_t[B])$ by Observation 6.1 (ii). Then, by Observation 6.1 (iv), we obtain $\mathrm{tw}(G_t[B]) \leqslant \mathrm{tw}(G_t[B - \{v_1, \dots, v_t\}]) + t$. As $B - \{v_1, \dots, v_t\}$ is an independent set in G_t, we obtain $\mathrm{tw}(G_t[B]) \leqslant t$. This yields $\mathrm{tw}(G_t) = \max\{t, \mathrm{tw}(G_t - \bigcup_{b\in B} V(G_b))\} = t$. ◊

Suppose for the sake of a contradiction that there is a family $(\mathbf{u})_{u\in V(G_t)}$ of vectors in $\mathbb{F}_2^{t+1}$ such that $\pi(yz) = \mathbf{y} \cdot \mathbf{z}$ for every edge yz of G_t.

Claim 6.7. — *The vectors $(\mathbf{b})_{b\in B}$ are pairwise distinct.*

Proof of the claim. First consider some distinct $\mathbf{x}, \mathbf{x}' \in \mathbb{F}_2^t$. Then there exists some $i \in [t]$ such that $\mathbf{x}_i \neq \mathbf{x}'_i$. If $\mathbf{w}_{\mathbf{x}} = \mathbf{w}_{\mathbf{x}'}$, we obtain $\pi(w_{\mathbf{x}}v_i) = \mathbf{w}_{\mathbf{x}}\mathbf{v}_i = \mathbf{w}_{\mathbf{x}'}\mathbf{v}_i = \pi(w_{\mathbf{x}'}v_i)$, a contradiction to the construction.

Now consider some distinct $i, j \in [t]$ and some $\mathbf{x} \in \mathbb{F}_2^t$ with $\mathbf{x}_i \neq \mathbf{x}_j$. If $\mathbf{v}_i = \mathbf{v}_j$, we obtain $\pi(w_{\mathbf{x}}v_i) = \mathbf{w}_{\mathbf{x}}\mathbf{v}_i = \mathbf{w}_{\mathbf{x}}\mathbf{v}_j = \pi(w_{\mathbf{x}}v_j)$, a contradiction to the construction.

Finally consider some $i \in [t]$ and $\mathbf{x} \in \mathbb{F}_2^t$. If $\mathbf{v}_i = \mathbf{w}_{\mathbf{x}}$, we obtain $\pi(v_i a_{i,\mathbf{x}}) = \mathbf{v}_i\mathbf{a}_{i,\mathbf{x}} = \mathbf{w}_{\mathbf{x}}\mathbf{a}_{i,\mathbf{x}} = \pi(w_x a_{i,\mathbf{x}})$, a contradiction to the construction. ◊

As $|B| = 2^t + t > 2^t + 1$, by Claim 6.7 and by Lemma 6.4, there exists $b \in B$ such that $\mathbf{b}^\perp$ has an orthonormal basis $\mathbf{x}_1, \dots, \mathbf{x}_t$. Now, for every $z \in V(G_b)$, we have $\mathbf{z} \in \mathbf{b}^\perp$ and so there are scalars $z_1, \dots, z_t$ in $\mathbb{F}_2$ such that $\mathbf{z} = \sum_{i=1}^t z_i\,\mathbf{x}_i$. Let $\mathbf{z}' = (z_1, \dots, z_t) \in \mathbb{F}_2^t$ for every $z \in V(G_b)$. Then, as $\mathbf{x}_1, \dots, \mathbf{x}_t$ is an orthonormal basis of $\mathbf{b}^\perp$, for every edge $yz \in E(G_b)$, we

have

$$\begin{aligned}\mathbf{y}' \cdot \mathbf{z}' &= \sum_{i\in[t]} y_i z_i \\ &= \sum_{i,j\in[t],i\neq j} y_i z_j \ \mathbf{x}_i \cdot \mathbf{x}_j + \sum_{i\in[t]} y_i z_i \ \mathbf{x}_i \cdot \mathbf{x}_i \\ &= \sum_{i\in[t]} y_i \mathbf{x}_i \cdot \sum_{j\in[t]} z_j \mathbf{x}_j \\ &= \mathbf{y} \cdot \mathbf{z} = \pi(yz).\end{aligned}$$

This contradicts the assumption on G_{t-1}.

Thus, by Observation 2.4, we have $\operatorname{diam}(\mathcal{I}(G_t)) > t+1$. Since G_t has treewidth t by Claim 6.6, this proves the theorem. □

7. K_t-minor-free and planar graphs

This section is dedicated to proving the results guaranteeing small inversion diameter for graphs contained in certain minor-closed classes. It was proved by van den Heuvel et al. [10] that for every $t \geqslant 4$ and every K_t-minor-free graph G, we have $\chi_s(G) \leqslant 5\binom{t}{2}(t-3)$. Together with Corollary 4.6, this yields the following result.

Corollary 7.1. — *For every proper minor-closed class $\mathcal{G}$ of graphs, there exists a constant $\alpha_{\mathcal{G}}$ such that* $\operatorname{diam}(\mathcal{I}(G)) \leqslant \alpha_{\mathcal{G}}$ *for every $G \in \mathcal{G}$.*

Hence proper minor-closed families have bounded inversion diameter. In the remainder of this section, we prove more precise bounds for some particular minor-closed classes.

7.1. K_t-minor-free graphs

In this section, we deal with K_t-minor-free graphs. To do so, we will use known bounds on strong colouring numbers. By the *length* of a path, we refer to its number of edges. Given a graph G, an ordering $<$ of its vertices, an integer $r \geqslant 0$, and $u, v \in V(G)$, we say that v is *r-strongly reachable from u in G for $<$* if there is a path P from u to v of length at most r such that $V(P) \cap \{w \in V(G) \mid w \leqslant u\} = \{u, v\}$. Note that u is r-strongly reachable from u. We denote by $\operatorname{SReach}_r[G, <, u]$ the set of all vertices in G r-strongly reachable from u in G for $<$.

THEOREM 7.2 ([10]). — *For every integer $t \geqslant 3$ and for every K_t-minor-free graph G, there is an ordering $<$ of $V(G)$ such that*

$$|\mathrm{SReach}_r[G, <, u]| \leqslant \binom{t-1}{2}(2r+1)$$

for every nonnegative integer r and every vertex $u \in V(G)$.

Note that in [10], this theorem is stated without specifying that $<$ does not depend on r. However, this fact follows from the proof.

Together with Theorem 3.3, Theorem 7.2 implies the following.

COROLLARY 7.3. — *Let t be an integer with $t \geqslant 3$. For every K_t-minor-free graph G,*

$$\mathrm{diam}(\mathcal{I}(G)) \leqslant 8\binom{t-1}{2}.$$

Proof. — Let $<$ be an ordering of $V(G)$ as in Theorem 7.2. For every vertex u of G, we have

(1) $|N_<(u)| \leqslant 3\binom{t-1}{2} - 1$, and

(2) $|\{X \subseteq V(G) \mid \exists\, v \in N(u), u < v, X \subseteq \{w \in N(v) \mid w < u\}\}| \leqslant 2^{5\binom{t-1}{2}}$.

Hence G is strongly $8\binom{t-1}{2}$-degenerate, and so $\mathrm{diam}(\mathcal{I}(G)) \leqslant 8\binom{t-1}{2}$ by Theorem 3.3. □

7.2. Planar graphs

7.2.1. General bound

A planar graph is K_5-minor-free, so by Corollary 7.3, its inversion diameter is at most 48. We shall improve on this bound by showing that $\mathrm{diam}(\mathcal{I}(G)) \leqslant 12$ for every planar graph G. This bound is obtained by using the following result establishing the existence of an ordering $<$ such that $\mathrm{SReach}_r[G, <, u]$ has small size for every u.[(3)]

THEOREM 7.4 ([10]). — *For every planar graph G, there is an ordering $<$ of $V(G)$ such that*

$$|\mathrm{SReach}_r[G, <, u]| \leqslant 5r + 1$$

for every vertex $u \in V(G)$ and every nonnegative integer r.

COROLLARY 7.5. — $\mathrm{diam}(\mathcal{I}(G)) \leqslant 12$ *for every planar graph G.*

[(3)] As in Theorem 7.2, the statement in [10] of Theorem 7.4 does not specify that the ordering $<$ is the same for every r. However, this fact follows from the proof.

Proof. — Let $<$ be an ordering of $V(G)$ as in Theorem 7.4. Let $u \in V(G)$. Let $A = \{u\} \cup \bigcup_{v \in N_>(u)} N_{<u}(v) \subseteq \mathrm{SReach}_2[G, <, u]$ and $B = N_>(u)$. Consider the bipartite graph H with bipartition (A, B) and with edges inherited from G. Let $H' = (A, B', E')$ be the graph obtained from H by removing vertices of degree at most 2 in B. Observe that H' is a subgraph of G and hence planar. Moreover, as H' is bipartite, we obtain that the girth of H' is at least 4. It hence follows from Euler's formula that $m(H') \leqslant 2(|A| + |B'|) - 4$. Moreover, every vertex in B' has degree at most 5, and has u as a neighbour. For $i \in \{2, 3, 4\}$, let d_i be the number of vertices in B' of degree $i + 1$ in H'. Then we have

$$\begin{aligned}|\{X \subseteq V(G) \mid \exists\, v \in N(u), u < v, X \subseteq N_{<u}(v)\}| \\ \leqslant 1 + \binom{10}{1} + \binom{2}{2} d_2 + \binom{3}{2 \leqslant \cdot \leqslant 3} d_3 + \binom{4}{2 \leqslant \cdot \leqslant 4} d_4\end{aligned}$$

using the notation $\binom{n}{k_1 \leqslant \cdot \leqslant k_2} = \sum_{k_1 \leqslant k \leqslant k_2} \binom{n}{k}$. Moreover,

$$3d_2 + 4d_3 + 5d_4 \leqslant m(H') \leqslant 2(|A| + |B'|) - 4 \leqslant 2(11 + d_2 + d_3 + d_4) - 4.$$

We deduce

$$d_2 + 2d_3 + 3d_4 \leqslant 18$$

and so

$$d_2 + 4d_3 + 11d_4 \leqslant \frac{11}{3}(d_2 + 2d_3 + 3d_4) \leqslant 66.$$

Finally we have

$$|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}| \leqslant 11 + d_2 + 4d_3 + 11d_4 \leqslant 77.$$

Therefore,

(1) $|N_<(u)| \leqslant 5$ and
(2) $|\{X \subseteq V(G) \mid \exists\, v \in N_>(u), X \subseteq N_{<u}(v)\}| \leqslant 77 < 2^7$,

and it follows that G is strongly 12-degenerate and so $\mathrm{diam}(\mathcal{I}(G)) \leqslant 12$ by Theorem 3.3. □

For the lower bound on the maximum inversion diameter of a planar graph, we executed a computer search[(4)]. As a result, we found a planar graph whose inversion diameter is at least 5.

Proposition 7.6. — *If G if the planar graph depicted on Figure 7.1, then* $\mathrm{diam}(\mathcal{I}(G)) \geqslant 5$.

[(4)] The source code of our program is available at https://github.com/ClementRambaud/inversion-diameter. We also used the program `nauty` [13] (available at https://users.cecs.anu.edu.au/~bdm/nauty/) and `plantri` (available at https://users.cecs.anu.edu.au/~bdm/plantri/). The program `GNU parallel` [17] has also been used.

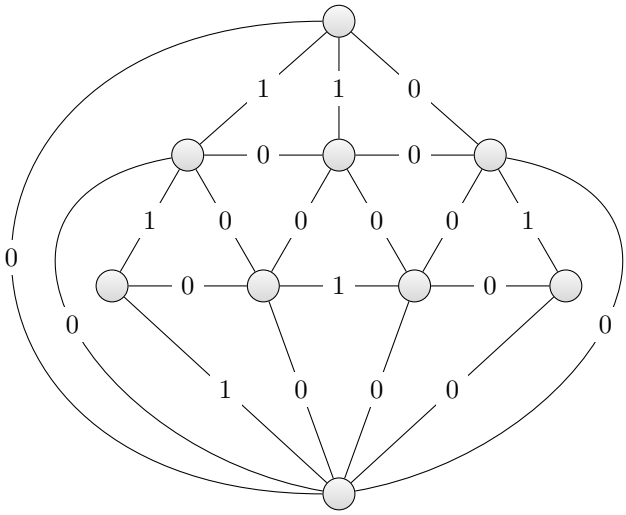


Figure 7.1. A planar graph G with inversion diameter at least 5. More precisely, if $\pi\colon E(G) \to \mathbb{F}_2$ is the labelling function depicted, then there is no family $(\mathbf{u})_{u\in V(G)} \in \mathbb{F}_2^4$ such that $\pi(uv) = \mathbf{u}\cdot\mathbf{v}$ for every $uv \in E(G)$.

7.2.2. Planar graphs of large girth

In this section, we show that planar graphs of large enough girth have inversion diameter at most 3, and that this bound it tight.[(5)]

THEOREM 7.7. — *Let G be a planar graph. If G has girth at least 8, then* $\operatorname{diam}(\mathcal{I}(G)) \leqslant 3$.

The tightness of this result is shown by the following proposition.

PROPOSITION 7.8. — *For every even positive integer g, there is a bipartite planar graph G with girth g such that* $\operatorname{diam}(\mathcal{I}(G)) \geqslant 3$.

Proof. — Let G be the graph obtained from a cycle on g vertices $C = v_1v_2\cdots v_g$ by adding for every vertex $v \in V(C)$ a new vertex w_v adjacent uniquely to v. By construction, we have $d_G(v) \geqslant 3$ for all $v \in V(C)$. We hence obtain $\operatorname{diam}(\mathcal{I}(G)) \geqslant 3$ by Lemma 5.11. □

(5) In the longer version of the paper, available at https://arxiv.org/pdf/2405.04119v2, upper bounds on the inversion diameter of planar graph G with girth g smaller than 8 can be found: if $g = 7$ then $\operatorname{diam}(\mathcal{I}(G)) \leqslant 6$, and if $g = 5, 6$ then $\operatorname{diam}(\mathcal{I}(G)) \leqslant 10$.

In fact, we shall prove a result on graphs of maximum average degree at most $2+\frac{8}{11}$ which implies Theorem 7.7 using the following well-known bound on the maximum average degree of planar graphs of large girth.

LEMMA 7.9 (Folklore). — *If G is a planar graph with girth at least g, then* $\mathrm{Mad}(G) < 2+\frac{4}{g-2}$.

The proof of this lemma, which is a direct application of Euler's formula, is left to the reader. By Lemma 7.9, in order to prove Theorem 7.7, it is sufficient to prove that if $\mathrm{Mad}(G) < 2+\frac{2}{3}$ then $\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$. We shall in fact prove the following stronger result.

THEOREM 7.10. — *Let G be a graph. If* $\mathrm{Mad}(G) \leqslant 2+\frac{8}{11}$, *then* $\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$.

Let G be a graph, and k a positive integer. A *k-vertex* (resp. *($\geqslant k$)-vertex*, *($\leqslant k$)-vertex*) is a vertex with degree k (resp. at least k, at most k).

Let G be a graph and let $\pi\colon E(G) \to \mathbb{F}_2$. For every positive integer d, a *(G,π)-realisation* of dimension d is a family $(\mathbf{u})_{u\in V(G)}$ of vectors in $\mathbb{F}_2^d$ such that for every edge uv of G, we have $\pi(uv) = \mathbf{u}\cdot\mathbf{v}$. A (G,π)-realisation is *strict* if $\mathbf{u} \neq \mathbf{0}$ for every $u \in V(G)$. Hence, Observation 2.4 can be rephrased as follows: $\mathrm{diam}(\mathcal{I}(G)) \leqslant d$ if and only if for every $\pi\colon E(G) \to \mathbb{F}_2$, there is a (G,π)-realisation of dimension d.

If H is a subgraph of G, we abbreviate an $(H, \pi|_{E(H)})$-realisation to an *(H,π)-realisation.* We further say that a (G,π)-realisation *extends* an (H,π)-realisation if the two realisations coincide on $V(H)$.

The proof of Theorem 7.10 uses the Discharging Method. However, instead of choosing a minimum counterexample to our main result, we choose a minimum counterexample to the slightly stronger result that no strict (G,π)-realisation exists for the given instance. We first present a collection of structural properties. As some of these results need to be proved in a slightly stronger form in order to be reused later, we give them in a form describing when certain realisations can be extended rather than explicitly speaking about a minimum counterexample. These structural results are contained in Lemmas 7.11 to 7.16. Then, we conclude by giving the proof of Theorem 7.10 in the form of a discharging procedure.

We first deal with 1-vertices.

LEMMA 7.11. — *Let G be a graph, let $\pi\colon E(G) \to \mathbb{F}_2$, and let v be a 1-vertex in G. Any strict $(G-v,\pi)$-realisation of dimension 3 can be extended into three different strict (G,π)-realisations of dimension 3.*

Proof. — Let u be the unique neighbour of v. Since $\mathbf{u} \neq \mathbf{0}$, the solutions $\mathbf{v}$ of the equation $\mathbf{u}\cdot\mathbf{v} = \pi(uv)$ form an affine space of dimension $3-1=2$.

Thus, there are at least $2^2 = 4$ vectors $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = \pi(uv)$. Since at most one of them is $\mathbf{0}$, we obtain at least 3 extensions. □

We now establish that the configurations for which a realization fails to extend to a 2-vertex are very restricted.

LEMMA 7.12. — *Let G be a graph, let $\pi\colon E(G) \to \mathbb{F}_2$, and let v be a 2-vertex in G with neighbours u_1 and u_2. A strict $(G - v, \pi)$-realisation $(\mathbf{u})_{u \in V(G-v)}$ of dimension 3 can be extended into a strict (G, π)-realisation unless $\mathbf{u_1} = \mathbf{u_2}$ and $\pi(u_1 v) \neq \pi(u_2 v)$.*

Proof. — If $\mathbf{u_1} \neq \mathbf{u_2}$, then $\mathbf{u_1}, \mathbf{u_2}$ are linearly independent. This implies that the affine space of the solutions $\mathbf{v}$ of

$$\begin{cases} \mathbf{u_1} \cdot \mathbf{v} = \pi(u_1 v) \\ \mathbf{u_2} \cdot \mathbf{v} = \pi(u_2 v) \end{cases}$$

is nonempty and has dimension $3 - 2 = 1$. Hence there is a solution $\mathbf{v}$ which is not $\mathbf{0}$.

Now suppose $\mathbf{u_1} = \mathbf{u_2}$ and $\pi(u_1 v) = \pi(u_2 v)$. Since $\mathbf{u_1} \neq \mathbf{0}$, there is a vector $\mathbf{v}$ such that $\mathbf{u_1} \cdot \mathbf{v} = \mathbf{u_2} \cdot \mathbf{v} = \pi(u_1 v) = \pi(u v_2)$. This proves the lemma. □

We are now ready to exclude the existence of two adjacent 2-vertices in a minimum counterexample.

LEMMA 7.13. — *Let G be a graph, $\pi\colon E(G) \to \mathbb{F}_2$, and v_1, v_2 two adjacent 2-vertices in G. Any strict $(G - \{v_1, v_2\}, \pi)$-realisation of dimension 3 can be extended into a strict (G, π)-realisation.*

Proof. — Let $(\mathbf{u})_{u \in V(G - \{v_1, v_2\})}$ be a $(G - \{v_1, v_2\}, \pi)$-realisation of dimension 3. Let u_1 (resp. u_2) be the neighbour of v_1 (resp. v_2) distinct from v_2 (resp. v_1). By Lemma 7.11, there are three possible non-zero vectors $\mathbf{v}_1$ that extend the realisation $(\mathbf{u})_{u \in V(G - \{v_1, v_2\})}$ to $G - v_2$. So one can assign $\mathbf{v}_1$ so that $\mathbf{v}_1 \neq \mathbf{u}_2$. Then, by Lemma 7.12, there is a choice of $\mathbf{v}_2$ that extends the realisation to G. □

We use $N[v]$ to denote the closed neighbourhood of v, that is, the set containing v and all its neighbours.

The next result shows that, in a minimum counterexample, every vertex that is adjacent only to 2-vertices is of degree at least 7.

LEMMA 7.14. — *Let G be a graph, let $\pi\colon E(G) \to \mathbb{F}_2$, and let v be a d-vertex, for some $d \leqslant 6$, adjacent to d 2-vertices in G. Any strict $(G - N[v], \pi)$-realisation of dimension 3 can be extended into a strict (G, π)-realisation of dimension 3.*

Proof. — Let $N(v) = \{u_1, \dots, u_d\}$ and, for all $i \in [d]$, let t_i be the neighbour of u_i distinct from v. Choose $\mathbf{v} \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$ such that $\mathbf{v} \notin \{\mathbf{t_1}, \dots, \mathbf{t_d}\}$. This is possible because there are seven non-zero vectors in $\mathbb{F}_2^3$ and $d \leqslant 6$. Then, by Lemma 7.12, there is a choice of $\mathbf{u}_1, \dots, \mathbf{u}_d$ that extends the realisation $(\mathbf{u})_{u \in V(G-N(u))}$ to G. □

We now turn to the 3-vertices. We first deal with the case that a 3-vertex is adjacent to two 2-vertices.

Lemma 7.15. — *Let G be a graph, $\pi\colon E(G) \to \mathbb{F}_2$, and v be a 3-vertex adjacent to two 2-vertices u_1 and u_2 in G. Any strict $(G - \{v, u_1, u_2\}, \pi)$-realisation $(\mathbf{u})_{u \in V(G-\{v,u_1,u_2\})}$ of dimension 3 can be extended into a strict (G, π)-realisation of dimension 3.*

Proof. — Let t_1 (resp. t_2) be the neighbour of u_1 (resp. u_2) distinct from v. By Lemma 7.11, there are three possible non-zero vectors $\mathbf{v}$ in $\mathbb{F}_2^3$ with which we can extend the realisation to $G - \{u_1, u_2\}$. So one can choose $\mathbf{v}$ among them so that $\mathbf{v} \notin \{\mathbf{t_1}, \mathbf{t_2}\}$. Then, by Lemma 7.12, there is a choice of $\mathbf{u}_1, \mathbf{u}_2$ that extends the realisation $(\mathbf{u})_{u \in V(G-\{u_1,u_2\})}$ to G. □

A *deficient vertex* is a 3-vertex having one 2-neighbour. As indicated by Lemma 7.15, deficient 3-vertices are in some way critical for our discharging procedure. We give one more structural result that restricts the existence of deficient vertices.

Lemma 7.16. — *Let G be a graph, let $\pi\colon E(G) \to \mathbb{F}_2$. Let y be a deficient vertex adjacent to two deficient vertices x and z in G, and let x', y', z' be the 2-neighbours of x, y, z respectively.*

Any strict $(G - \{x, y, z, x', y', z'\}, \pi)$-realisation of dimension 3 can be extended into a strict (G, π)-realisation of dimension 3.

Proof. — Let x'' (resp. y'', z'') be the neighbour of x' (resp. y', z') distinct from x (resp. y, z).

By Lemma 7.12, there are three possibilities to extend $(\mathbf{u})_{u \in V(G-\{x,y,z,x',y',z'\})}$ to x, and so at least two of them are different from $\mathbf{x}''$. Let I_x be the set of these vectors. Similarly, let I_z be the set of possible extensions of $(\mathbf{u})_{u \in V(G-\{x,y,z,x',y',z'\})}$ to z different from $\mathbf{z}''$.

First suppose $\pi(xy) = 1$. Then choose $\mathbf{x} \in I_x$ and $\mathbf{z} \in I_z$ such that $\mathbf{x} \neq \mathbf{z}$. Then the affine space of the common solutions $\mathbf{y}$ of the equations $\mathbf{y} \cdot \mathbf{x} = \pi(xy)$ and $\mathbf{y} \cdot \mathbf{z} = \pi(yz)$ has dimension 1 and does not contain $\mathbf{0}$ (since $\pi(xy) \neq 0$). Then choose $\mathbf{y}$ among these solutions such that $\mathbf{y} \neq \mathbf{y}''$. Finally, extend the realisation to x', y', z' using Lemma 7.12. We proceed symmetrically if $\pi(yz) = 1$. Now suppose $\pi(xy) = \pi(yz) = 0$.

Suppose $I_x \cap I_z \neq \emptyset$. Then take $\mathbf{x} = \mathbf{z} \in I_x \cap I_z$. The solutions $\mathbf{y}$ of the equation $\mathbf{y} \cdot \mathbf{x} = 0$ form a space of dimension $3 - 1 = 2$. Hence there are $2^2 = 4$ such solutions. Then take for $\mathbf{y}$ one of these solutions distinct from $\mathbf{y}''$ and from $\mathbf{0}$. Then we can extend the realisation to x', y', z' by Lemma 7.12. Now we assume $I_x \cap I_z = \emptyset$.

Let $I_x = \{\mathbf{x}_1, \mathbf{x}_2\}$ and $I_z = \{\mathbf{z}_1, \mathbf{z}_2\}$. For every $i, j \in \{1, 2\}$, the choice of $\mathbf{x} = \mathbf{x}_i$ and $\mathbf{z} = \mathbf{z}_j$ can not be extended only if $\{\mathbf{0}, \mathbf{y}''\} = \{\mathbf{x}_i, \mathbf{z}_j\}^\perp$. In particular, $\{\mathbf{x}_1, \mathbf{z}_1\}$ and $\{\mathbf{x}_1, \mathbf{z}_2\}$ span the same vector space, so $\{\mathbf{x}_1, \mathbf{z}_1, \mathbf{z}_2\}$ is linearly dependent. As $\mathbf{x}_1, \mathbf{z}_1$, and $\mathbf{z}_2$ are pairwise distinct, we obtain $\mathbf{x}_1 + \mathbf{z}_1 + \mathbf{z}_2 = 0$. A similar argument shows that $\mathbf{x}_2 + \mathbf{z}_1 + \mathbf{z}_2 = 0$. It follows that $\mathbf{x}_1 = \mathbf{x}_2$, a contradiction.

This concludes the proof of the lemma. □

Proof of Theorem 7.10. — We will show that for every graph with $\mathrm{Mad}(G) < 2 + \frac{8}{11}$ and for every function $\pi \colon E(G) \to \mathbb{F}_2$, there is a family of non-zero vectors $(\mathbf{u})_{u \in V(G)}$ in $\mathbb{F}_2^3$ such that $\pi(uv) = \mathbf{u} \cdot \mathbf{v}$ for every edge uv of G, that is, there exists a strict (G, π)-realisation of dimension 3. By Observation 2.4, this will imply $\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$, and so prove the theorem. Let G be a counterexample to this statement with $|V(G)|$ minimum. The above lemmas yield the following properties.

(P1) $\delta(G) \geqslant 2$ (Lemma 7.11).
(P2) Two 2-vertices are not adjacent (Lemma 7.13).
(P3) A 3-vertex has at most one 2-neighbour (Lemma 7.15).
(P4) Every 4-, 5-, or 6-vertex has at least one ($\geqslant 3$)-neighbour (Lemma 7.14).
(P5) A deficient vertex is adjacent to at most one deficient vertex (Lemma 7.16).

We shall now use the Discharging Method. The initial charge of each vertex v is its degree $d_G(v)$. We now apply the following discharging rules.

(R1) Every ($\geqslant 3$)-vertex sends $\frac{4}{11}$ to its 2-neighbours.
(R2) Every non-deficient ($\geqslant 3$)-vertex sends $\frac{1}{11}$ to its deficient neighbours.

Let us examine the final charge $w(v)$ of a vertex v in $V(G)$. By (P1), we have $d_G(v) \geqslant 2$.

- If v is a 2-vertex, then it has two ($\geqslant 3$)-neighbours by (P2). It receives $\frac{4}{11}$ from each of them, so $w(v) = 2 + 2 \times 4/11 = 2 + 8/11$.
- If v is a deficient 3-vertex, then it has one 2-neighbour and has at least one non-deficient ($\geqslant 3$)-neighbour by (P5).

Thus it sends $\frac{4}{11}$ to its 2-neighbour and receives at least $\frac{1}{11}$ from at least one non-deficient neighbour. So $w(v) \geqslant 3 - 4/11 + 1/11 = 2 + 8/11$.

- If v is a non-deficient 3-vertex, then its sends $1/11$ to its at most three deficient neighbours. So $w(v) \geqslant 3 - 3 \times 1/11 = 2 + 8/11$.
- If v is a 4-vertex, then it sends $4/11$ to each of its 2-neighbours by (R1) and $1/11$ to each of its deficient neighbours by (R2). By (P4), v has at most three 2-neighbours. So $w(v) \geqslant 4 - 3 \times 4/11 - 1/11 = 2 + 9/11$.
- If v is a ($\geqslant 5$)-vertex, then it sends at most $4/11$ to each of its neighbours. So $w(v) \geqslant d(v) - d(v) \times 4/11 = d(v) \times 7/11 \geqslant 35/11 > 2 + 8/11$.

Hence $w(v) \geqslant 2+8/11$ for every vertex v. Consequently, $\sum_{v \in V(G)} d(v) = \sum_{v \in V(G)} w(v) \geqslant (2 + 8/11)|V(G)|$. We conclude that $\mathrm{Mad}(G) \geqslant \mathrm{Ad}(G) \geqslant 2 + 8/11$, a contradiction. □

8. Complexity

In this section, we consider the computational complexity of the following two related problems.

k-Inversion-Distance
Instance: Two orientations $\vec{G}_1$ and $\vec{G}_2$ of a graph G.
Question: Are $\vec{G}_1$ and $\vec{G}_2$ at distance at most k in $\mathcal{I}(G)$?

k-Inversion-Diameter
Instance: A graph G.
Question: Does $\mathcal{I}(G)$ have diameter at most k ?

0-Inversion-Distance is trivial as two orientations of some graph are at distance 0 if and only if they are the same. Furthermore, 1-Inversion-Distance can be solved in polynomial time because of the following easy characterisation of orientations at distance 1.

Proposition 8.1. — *Let $\vec{G}_1$ and $\vec{G}_2$ be two orientations of a graph G. There exists X such that $\vec{G}_2 = \mathrm{Inv}(\vec{G}_1; X)$ if and only if every edge of $E_=$ has at most one end-vertex incident to an edge of $E_{\neq}$.*

Proposition 4.3 implies that 0-Inversion-Diameter and 1-Inversion-Diameter can be solved in polynomial time.

The aim of this section is to prove that k-Inversion-Diameter and k-Inversion-Distance are NP-hard for all $k \geqslant 2$.

We need some preliminaries. The first one is the following well-known lemma, which can be easily proved using a well-known theorem of Euler, see [6].

LEMMA 8.2 (Folklore). — *Every graph G admits an orientation $\vec{G}$ such that every vertex v has in-degree at least $\lfloor d(v)/2 \rfloor$ in $\vec{G}$. Moreover, such an orientation can be computed in polynomial time.*

LEMMA 8.3. — *Let $d \geqslant 2$ and let $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d \setminus \{\mathbf{0}\}$ be two distinct vectors. For every $x, y \in \mathbb{F}_2$, there exists $\mathbf{w} \in \mathbb{F}_2^d$ such that $\mathbf{u} \cdot \mathbf{w} = x, \mathbf{v} \cdot \mathbf{w} = y$.*

Proof. — As $\mathbf{u} \neq \mathbf{v}$ are distinct non zero vectors in $\mathbb{F}_2^d$, the matrix M whose rows are u and v has rank 2. Hence its image is a 2-dimensional space, and so the mapping

$$\begin{pmatrix} \mathbb{F}_2^d \longrightarrow \mathbb{F}_2^2 \\ \mathbf{w} \longmapsto M \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}, \mathbf{v} \cdot \mathbf{w}) \end{pmatrix}$$

is surjective. □

For a graph G, we denote by $G^{(1)}$ the graph obtained from G by subdividing every edge of G exactly once. The following theorem links $\operatorname{diam}(\mathcal{I}(G^{(1)}))$ with $\chi(G)$. Note that similar results for χ_s, χ_a, χ_o have been proved by Wood [18].

THEOREM 8.4. — *For every graph G with $\delta(G) \geqslant 2$ and for every integer $k \geqslant 2$, one can construct in polynomial time two orientations O_1, O_2 of $G^{(1)}$ such that the following are equivalent:*

(i) $\operatorname{dist}_{\mathcal{I}(G^{(1)})}(O_1, O_2) \leqslant k$,
(ii) $\operatorname{diam}(\mathcal{I}(G^{(1)})) \leqslant k$,
(iii) $\chi(G) \leqslant 2^k - 1$.

Proof. — Let G be a graph and let $k \geqslant 2$ be an integer. We denote by $ux_{uv}v$ the path of length 2 replacing the edge $uv \in E(G)$ in $G^{(1)}$.

By Lemma 8.2, G has an orientation $\vec{G}$ such that every vertex has in-degree at least 1. For every arc $uv \in A(\vec{G})$, set $\pi_0(ux_{uv}) = 0$ and $\pi_0(x_{uv}v) = 1$. Let O_1 be an arbitrary orientation of $G^{(1)}$ and let O_2 be obtained from O_1 by reversing every edge $e \in E(G^{(1)})$ such that $\pi_0(e) = 1$.

We first show that (iii) implies (ii). Assume that $\chi(G) \leqslant 2^k - 1$. Then there exists a mapping $\phi\colon V(G) \to \mathbb{F}_2^k \setminus \{0\}$ such that $\phi(u) \neq \phi(v)$ for every $uv \in E(G)$. Consider a function $\pi\colon E(G^{(1)}) \to \mathbb{F}_2$. We will construct a family $(\mathbf{w})_{w \in V(G)}$ of vectors in $\mathbb{F}_2^k$ such that $\pi(ww') = \mathbf{w} \cdot \mathbf{w'}$ for every edge ww' of $G^{(1)}$. This will imply that $\operatorname{diam}(\mathcal{I}(G^{(1)})) \leqslant k$ by Observation 2.4. We take $\mathbf{u} = \phi(u)$ for every vertex $u \in V(G)$, and we choose $\mathbf{x}_{uv}$ such that

$\mathbf{u} \cdot \mathbf{x}_{uv} = \pi(ux_{uv})$ and $\mathbf{v} \cdot \mathbf{x}_{uv} = \pi(x_{uv}v)$. This is possible by Lemma 8.3. Hence (ii) holds.

Clearly, (ii) implies (i).

We finally show that (i) implies (iii). Suppose that $\mathrm{dist}_{\mathcal{I}(G^{(1)})}(O_1, O_2) \leqslant k$. Then, by the same argument as in the proof of Observation 2.4, there exists a family $(\mathbf{w})_{w \in V(G^{(1)})}$ of vectors in $\mathbb{F}_2^k$ such that $\pi_0(ww') = \mathbf{w} \cdot \mathbf{w}'$ for every edge ww' of $G^{(1)}$. Then for every edge uv of G, by the choice of π_0 we have $\mathbf{u} \cdot \mathbf{x}_{uv} \neq \mathbf{v} \cdot \mathbf{x}_{uv}$ and so $\mathbf{u} \neq \mathbf{v}$. Moreover, for every $u \in V(G)$, u is incident to an edge e in $G^{(1)}$ with $\pi_0(e) = 1$, which implies that $\mathbf{u} \neq \mathbf{0}$. Thus $u \mapsto \mathbf{u}$ is a proper $(2^k - 1)$-colouring of G, yielding (iii).

This concludes the proof of the theorem. □

It is well-known that deciding whether a graph G with $\delta(G) \geqslant 2$ has chromatic number at most k is NP-complete for any fixed integer $k \geqslant 3$ [11] and that deciding whether a planar graph G with $\delta(G) \geqslant 2$ has chromatic number at most 3 is NP-complete [8]. Note that for every graph G, $G^{(1)}$ has maximum average degree less than 4, has girth at least 6, and is bipartite. As a consequence, we obtain the following result.

Corollary 8.5. — *For every integer $k \geqslant 2$, k-Inversion-Diameter and k-Inversion-Distance are NP-hard, even when restricted to graphs G with $\mathrm{Mad}(G) < 4$. Moreover, 2-Inversion-Diameter and 2-Inversion-Distance are NP-hard even for planar bipartite graphs of girth at least 6.*

9. Conclusion and further work

In this paper, we established some bounds on the inversion diameter of graphs in various graph classes. We leave as open problems to improve several of them.

Problem 9.1. — *For every positive integer Δ, determine the maximum inversion diameter of a graph of maximum degree Δ.*

Theorem 2.5 and Theorem 5.1 imply that the answer lies between Δ and $2\Delta - 1$, and we conjectured in Conjecture 5.1 that it is actually Δ. This conjecture has been proved for $\Delta \leqslant 2$. So the first open case is for $k = 3$: we do not know whether it is true that $\mathrm{diam}(\mathcal{I}(G)) \leqslant 3$ for every subcubic graph G.

We also investigated the maximum inversion diameter of a graph of given treewidth. Our upper and lower bounds do not match and so we leave as an open problem to bridge this gap.

PROBLEM 9.2. — *For every positive integer t, determine the maximum inversion diameter of a graph of treewidth t.*

Theorem 6.2 and Theorem 6.5 imply that this maximum lies between $t+2$ and $2t$ for every $t \geqslant 2$.

Concerning K_t-minor-free graphs, we showed in Corollary 7.3 that they have inversion diameter in $\mathcal{O}(t^2)$, while Theorem 3.2 (and the classical fact that there are K_t-minor-free graphs with average degree in $\Omega(t\sqrt{\log t})$) imply that there are K_t-minor-free graphs with inversion diameter in $\Omega(t\sqrt{\log t})$. We leave as an open problem to bridge this gap.

PROBLEM 9.3. — *For every positive integer t, determine the maximum inversion diameter of a K_t-minor-free graph.*

Among minor-closed classes of graphs, planar graphs is arguably the most important one. This motivates the following problem.

PROBLEM 9.4. — *Determine the maximum inversion diameter over all planar graphs.*

Proposition 7.6 and Corollary 7.5 imply that it lies between 5 and 12.

We also believe that the upper bound in Theorem 2.5 can be generalised to the blow-up of any graph as follows.

CONJECTURE 9.5. — *For every graph G, for every positive integer t,*

$$\operatorname{diam}(\mathcal{I}(G[\overline{K}_t])) \leqslant t \cdot \operatorname{diam}(\mathcal{I}(G)).$$

This inequality would be tight for complete graphs by Theorem 2.5. Note that $\operatorname{diam}(\mathcal{I}(G[\overline{K}_t]))$ can be characterised as in Observation 2.4 using matrices over $\mathbb{F}_2$.

OBSERVATION 9.6. — *For every graph G, for every positive integers t, ℓ, the following are equivalent.*

(1) $\operatorname{diam}(\mathcal{I}(G[\overline{K}_t])) \leqslant \ell$.

(2) For every function $\Pi\colon E(G) \to \mathbb{F}_2^{t\times t}$, there is a family $(U)_{u\in V(G)}$ of matrices in $\mathbb{F}_2^{\ell\times t}$ such that

$$\Pi(uv) = U^\top \cdot V$$

for every edge $uv \in E(G)$.

Further, it would be interesting to understand how the density of the graph influences the diameter of its inversion graph. In particular, we are interested in the following question.

Problem 9.7. — *Is there an absolute constant α such that* $\operatorname{diam}(\mathcal{I}(G)) \leqslant \alpha\sqrt{|E(G)|}$ *holds for every graph G?*

This is motivated by the case of complete graphs, for which Theorem 2.5 implies that $\operatorname{diam}(\mathcal{I}(K_n)) = \Theta(\sqrt{|E(K_n)|})$.

Regarding complexity, we solved the most natural questions, but many remain. One of them is the following.

Problem 9.8. — *Is 2-Inversion-Diameter Π_2^{P}-hard?*

We refer the reader to [16] for some Π_2^{P}-complete problems. We could also study the complexity of the following dual problem.

k-Dual-Inversion-Diameter
Instance: A graph G.
Question: Does $\mathcal{I}(G)$ have diameter at least $|V(G)| - 1 - k$?

The first question is to determine whether it is NP-complete or polynomial-time solvable for every fixed k. Trivially, 0-Dual-Inversion-Diameter is poly- nomial-time solvable as the inversion diameter of a graph G is at most $|V(G)| - 1$. Since an edgeless graph has inversion diameter 0, Lemma 2.3 implies $\operatorname{diam}(\mathcal{I}(G)) \leqslant |V(G)| - \alpha(G)$ where $\alpha(G)$ denotes the maximum size of an independent set in G. Since every non-complete graph has an independent set of size 2, it follows that a graph G has inversion diameter at most $|V(G)| - 1$ if and only if it is complete. Thus 1-Dual-Inversion-Diameter is polynomial-time solvable.

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